FINITE TEMPERATURE FIELD THEORY

FINITE TEMPERATURE FIELD THEORY

Ashok Das

Department of Physics and Astronomy
University of Rochester
USA

World Scientific
Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 912805

USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Das, Ashok, 1953–

Finite temperature field theory / Ashok Das.

p. cm.

Includes bibliographical references and index.

ISBN 9810228562

1. Quantum field theory. 2. Gauge fields (Physics). 3. Mathematical physics.

QC174.45.D368 1997

530.14/3--dc21 97-8242

CIP

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

Copyright © 1997 by World Scientific Publishing Co. Pte. Ltd.

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the Publisher.

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

This book is printed on acid-free paper.

Printed in Singapore by Uto-Print

To

Kanjapa

and

Krushna

Introduction

In the past several years, there have been a lot of interest in various phenomena, in quantum field theories, at finite temperature. The formalism, for such field theoretic investigations, of course, dates back to Matsubara. However, over the years, there have been alternate methods of doing calculations in field theories at finite temperature which shed light on the structure of theories at finite temperature. And, as more and more studies are carried out in these formalisms, we become more and more aware of the subtleties at finite temperature and we understand the structure of these theories much better.

While many people are quite familiar with the Matsubara formalism or the imaginary time formalism, the alternate, real time formalisms are not as widely appreciated. However, the analysis of many phenomena of current interest need such alternate formalisms. Keeping this in mind, I have tried to describe all the three formalisms – imaginary time formalism, closed time path formalism and thermofield dynamics – right in the beginning chapters of this book. There are numerous examples that are worked out to give a sense of which formalism may be better suited for a given kind of problem. It is, of course, left completely to the reader to choose from among these formalisms, to carry out any calculation.

I also develop, in detail, various other concepts that are not as widely stressed in the literature. For example, the study of gauge theories at finite temperature, the cutting rules at finite temperature, the gauge dependence of the effective potential are just a few topics that are discussed at great lengths. Most of these topics are basic field theory questions and, therefore, to keep the material self-complete, many of the zero temperature topics on field theory are also worked out in detail.

The idea here is to bring out, from these finite temperature calculations, the similarities with and differences from what we know at zero temperature. Although, I had originally planned to include a discussion of the hard thermal loops in this book, neither space nor time allowed this. Moreover, I recognize that there exist, by now, several excellent sources of discussion of the hard thermal loops in the literature. The references, at the end of each chapter, only represent those which I have found useful in presenting the material. There are many more papers on the various topics being discussed and I apologize for not being able to include an exhaustive list of references.

The material in the first few chapters of this book were presented as a series of lectures at the Mehta Research Institute, Allahabad and I would like to thank Prof. H.S. Mani for his hospitality. I have benefited and learned a lot from all of my collaborators on the subject – K.S. Babu, P.F. Bedaque, M. Hott, A. Karev, S. Naik, S. Panda, P. Panigrahi and S. Vokos – and I would like to thank them all. I also acknowledge numerous suggestions (and help in simplifying many difficult derivations) from Drs. P.F. Bedaque and S. Pernice as well as from Profs. S. Okubo and H.A. Weldon. I would also like to thank M. Begel and T. Guptill for much needed assistance on computer related matters. All the figures in this book were drawn using PSTricks.

I appreciate all the help that many of my friends have provided during this project. But, most of all, I would like to thank my family members, especially Jhilli, for their support and understanding, particularly during the last part of the project.

Ashok Das

Rochester

Contents

1 Imaginary Time Formalism **1**

1.1 Introduction 1

1.2 Matsubara Formalism 4

1.3 Matsubara Frequencies 10

1.4 Path Integral Formulation 14

1.5 Applications 18

1.6 One Loop Mass Correction 19

1.7 Self-Energy at Finite Temperature 22

1.8 Covariant Description 32

1.9 References 37

2 Real Time Formalism **39**

2.1 Closed Time Path Formalism 39

2.2 Propagators 47

2.3 Matrix Structure for Propagators 51

2.4 Analyticity Properties 61

2.5 One Loop Propagator 64

2.6 References 67

3 Thermofield Dynamics **69**

3.1 General Formalism 70
3.2 Fermionic Oscillator 73
3.3 Bosonic Oscillator 79
3.4 Free Schrödinger Field Theory 83
3.5 Free Klein-Gordon Theory 90
3.6 KMS Condition 95
3.7 Dispersion Relations 96
3.8 Goldstone Theorem 99
3.9 References 103

4 Gauge Theories **105**

4.1 Gauge Theories at Zero Temperature 106
4.2 BRST Invariance 112
4.3 Ward Identities 120
4.4 Unitarity 125
4.5 Fermionic Boundary Conditions 133
4.6 Partition Function 138
4.7 Ward Identities at $T \neq 0$ 143
4.8 References 147

5 Cutting Rules at $T \neq 0$ **149**

5.1 Zero Temperature 150
5.2 Finite Temperature 167
5.3 Example 187
5.4 References 190

6 Symmetry Breaking **191**

6.1 Global Symmetry 192

6.2 Local Symmetry 199
6.3 Effective Potential 205
6.4 One Loop Potential 212
6.5 Symmetry Restoration 221
6.6 Dynamical Symmetry Breaking 226
6.7 References 230

7 Nielsen Identities **233**
7.1 Gauge Problem 234
7.2 Identities at $T = 0$ 240
7.3 Verification of Identities 251
7.4 Identities at $T \neq 0$ 255
7.5 References 257

8 Subtleties at $T \neq 0$ **259**
8.1 Degenerate Electron Gas 261
8.2 Calculation with ϵ–Regularization 266
8.3 Feynman Parameterization 269
8.4 Modification of Feynman Formula 271
8.5 Catalysis at $T \neq 0$ 274
8.6 References 289

9 Supersymmetry at $T \neq 0$ **291**
9.1 Supersymmetry 291
9.2 Supersymmetric Oscillator 298
9.3 Interacting Theory 309
9.4 Nicolai Map 312
9.5 Goldstino States 318
9.6 NonSimply Connected SpaceTime 321

9.7 References 332

10 Derivative Expansion **333**

10.1 Derivative Expansion 334

10.2 Soluble Models 338

10.3 Derivative Expansion at $T \neq 0$ 347

10.4 Anomaly at $T \neq 0$ 352

10.5 References 359

11 Non-Equilibrium Phenomena **361**

11.1 Disoriented Chiral Condensates 362

11.2 Free Klein-Gordon Theory 368

11.3 Nambu-Jona-Lasinio Model 377

11.4 Domain Growth 386

11.5 References 397

Index **399**

Chapter 1

Imaginary Time Formalism

1.1 Introduction

Let me begin by reviewing some of the general concepts from equilibrium statistical thermodynamics. The statistical behavior of a quantum system, in thermal equilibrium, is normally studied through an appropriate ensemble. In general, one defines a density matrix for the system as

$$\rho(\beta) = \mathrm{e}^{-\beta\mathcal{H}} \tag{1.1}$$

where β represents the inverse of the equilibrium temperature (We are assuming that the Boltzmann constant $k = 1$, otherwise, $\beta = \frac{1}{kT}$.), and $\mathcal{H}$ can be thought of as the appropriate Hamiltonian for the particu-

lar choice of the ensemble. Thus, for example, for a grand canonical ensemble, we have

$$\mathcal{H} = H - \mu N$$

where H and N represent, respectively, the dynamical Hamiltonian and the number operators for the particular system under study and μ corresponds to the chemical potential. For a canonical ensemble, on the other hand, we have

$$\mathcal{H} = H$$

Namely, we can think of the canonical ensemble as a grand canonical ensemble with a vanishing chemical potential. In our general discussions, we will keep the choice of the ensemble arbitrary, since the qualitative properties of a system at finite temperature do not depend on the nature of the ensemble. In working out specific examples, however, we will choose the ensemble that is appropriate for the particular physical system under study.

Given the density matrix, we can define the partition function of the system as

$$Z(\beta) = \mathrm{Tr}\,\rho(\beta) = \mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}} \tag{1.2}$$

where "Tr" stands for the trace or the sum over the expectation values in any complete basis. The ensemble average of any observable A is, then, defined as

$$\langle A\rangle_\beta = Z^{-1}(\beta)\,\mathrm{Tr}\,\rho(\beta)A = \frac{\mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}}A}{\mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}}} \tag{1.3}$$

The thermal average of the correlation function of any two operators, A and B, with different coordinates can similarly be written as

$$\langle AB\rangle_\beta = Z^{-1}(\beta)\,\mathrm{Tr}\,\rho(\beta)AB \tag{1.4}$$

where we have suppressed the coordinate dependence of the operators.

Let us next note that, given an ensemble and a set of Schrödinger operators, we can define operators in a (modified) Heisenberg picture as follows. For any arbitrary Schrödinger operator A, we have the Heisenberg operator $A_H(t)$ defined as

$$A_H(t) = \mathrm{e}^{i\mathcal{H}t}A\mathrm{e}^{-i\mathcal{H}t} \tag{1.5}$$

It is clear now that, for a general thermal correlation function of two Heisenberg operators $A_H(t)$ and $B_H(t')$, we can write

$$\begin{aligned}
\langle A_H(t)B_H(t')\rangle_\beta &= Z^{-1}(\beta)\,\mathrm{Tr}\,\rho(\beta)A_H(t)B_H(t') \\
&= Z^{-1}(\beta)\,\mathrm{Tr}\,e^{-\beta\mathcal{H}}A_H(t)B_H(t') \\
&= Z^{-1}(\beta)\,\mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}}A_H(t)\mathrm{e}^{\beta\mathcal{H}}\mathrm{e}^{-\beta\mathcal{H}}B_H(t') \\
&= Z^{-1}(\beta)\,\mathrm{Tr}\,A_H(t+i\beta)\mathrm{e}^{-\beta\mathcal{H}}B_H(t') \\
&= Z^{-1}(\beta)\,\mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}}B_H(t')A_H(t+i\beta) \\
&= \langle B_H(t')A_H(t+i\beta)\rangle_\beta
\end{aligned} \tag{1.6}$$

Here we have used eq. (1.5) as well as the cyclicity properties of the trace operation. It is important to note that eq. (1.6) holds independent of the Grassmann parities of the operators A and B, namely, for both bosonic as well as fermionic operators. Relations of the type (1.6) are known as the Kubo-Martin-Schwinger (KMS) relations. Let us note, in particular, from eq. (1.6), that

$$\langle A_H(t)A_H(t')\rangle_\beta = \langle A_H(t')A_H(t+i\beta)\rangle_\beta \tag{1.7}$$

As we will see later, this is what leads to the periodicity and the anti-periodicity in various two point Greens functions at finite temperature. The KMS conditions are also extremely useful in studying the asymptotic behaviour of various matrix elements as well as dispersion relations at finite temperature. (It is worth emphasizing here that cyclicity of the trace may fail if the trace diverges. This, as we will see later in chapter 3, may be the case when there is a spontaneous breakdown of a symmetry.)

1.2 Matsubara Formalism

In general, the partition function of a statistical system cannot be evaluated exactly. From the form of eq. (1.2), we see that even a perturbative expansion in powers of a coupling constant seems formidable primarily because we have a sum over the expectation values in all possible states in the Hilbert space and there is an infinite number of such states in any quantum field theory. The Matsubara formalism provides a way of evaluating the partition function perturbatively using a diagrammatic method which is analogous to what is used in conventional quantum field theories at zero temperature. There are several ways to introduce the Matsubara formalism – the most popular being the path integral method. Let me, instead, describe the operatorial method first and then go to the path integral method. In either of the approaches, the crucial observation lies in the fact that the density matrix in eq. (1.1) has the form of a time evolution operator for negative imaginary times.

We note from the definition of the density matrix in eq. (1.1) that if the total Hamiltonian, H, of the system can be separated into a free part and an interaction part

$$H = H_0 + H' \tag{1.8}$$

then, we can write

$$\mathcal{H} = \mathcal{H}_0 + H'$$

In this case, the density matrix can also be written as

$$\rho(\beta) = e^{-\beta\mathcal{H}} = \rho_0(\beta)S(\beta) \tag{1.9}$$

where we have defined

$$\rho_0(\beta) \equiv \mathrm{e}^{-\beta\mathcal{H}_0} \tag{1.10}$$

so that we can also write ($\mathcal{H}_0$ and H' need not commute)

$$S(\beta) = \mathrm{e}^{\beta\mathcal{H}_0}\,\mathrm{e}^{-\beta\mathcal{H}} = \rho_0^{-1}(\beta)\rho(\beta) \tag{1.11}$$

The density matrix, as we can see from eqs. (1.9)-(1.10), satisfies the Bloch equation which can be thought of as its evolution equation with $0 \leq \tau \leq \beta$

$$\begin{aligned} \frac{\partial\rho_0(\tau)}{\partial\tau} &= -\mathcal{H}_0\,\rho_0(\tau) \\ \frac{\partial\rho(\tau)}{\partial\tau} &= -\mathcal{H}\rho(\tau) = -(\mathcal{H}_0 + H')\rho(\tau) \end{aligned} \tag{1.12}$$

From equations (1.11)-(1.12), it follows, then, that the evolution equation satisfied by $S(\tau)$ is given by

$$\begin{aligned}\frac{\partial S(\tau)}{\partial \tau} &= \frac{\partial \rho_0^{-1}(\tau)}{\partial \tau}\rho(\tau) + \rho_0^{-1}(\tau)\frac{\partial \rho(\tau)}{\partial \tau} \\ &= \rho_0^{-1}(\tau)\mathcal{H}_0\rho(\tau) - \rho_0^{-1}(\tau)\mathcal{H}\rho(\tau) \\ &= \rho_0^{-1}(\tau)(\mathcal{H}_0 - \mathcal{H})\rho_0(\tau)\rho_0^{-1}(\tau)\rho(\tau) \\ &= -H_I'(\tau)S(\tau) \qquad (1.13)\end{aligned}$$

where we have defined

$$H_I'(\tau) = \rho_0^{-1}(\tau)H'\rho_0(\tau) = \mathrm{e}^{\tau\mathcal{H}_0}H'\mathrm{e}^{-\tau\mathcal{H}_0} \qquad (1.14)$$

We recognize the relations (1.13) and (1.14) from our studies of quantum field theories at zero temperature. Relation (1.14) simply defines a (modified) interaction picture through the relations

$$A_I(\tau) = \mathrm{e}^{\tau\mathcal{H}_0}A\mathrm{e}^{-\tau\mathcal{H}_0} \qquad (1.15)$$

Such a transformation is not necessarily unitary for real τ and, therefore, the adjoint of a transformed operator does not coincide with the transformed adjoint operator. One has to be careful in dealing with such operators. However, we note that if τ were a complex variable, then for imaginary values of τ, the transformation will be unitary. That is why the Matsubara formalism is also known as the imaginary time formalism. Another way of saying the same is to note that the operator implementing the evolution coincide with the usual ones (see, for example, eq. (1.5)) if we identify $t = -i\tau$, namely, on the negative imaginary time axis. (Sometimes questions come up such as whether a Hermitian field loses its hermiticity under such a rotation to imaginary time. The thing to note here is that the field remains Hermitian

with the appropriate definition of hermiticity for complex coordinates, namely, $\phi^\dagger(z) = \phi(z^*)$.)

We also recognize the evolution equation (1.13), from our studies of zero temperature field theory, as nothing other than the evolution equation for the time evolution operator in the interaction picture. Equation (1.13) can be formally integrated to give

$$S(\beta) = P_\tau \left(\mathrm{e}^{-\int_0^\beta d\tau H_I'(\tau)} \right) \tag{1.16}$$

where P_τ stands for ordering in the τ variable (and, we use this notation as opposed to the more conventional T_τ to avoid any possible confusion with "Tr"). This is, of course, quite analogous to the expression for the S-matrix in zero temperature field theory except for the fact that the time integration is over a finite interval along the imaginary axis. As in the zero temperature case, we can expand the exponential and each term in the expansion would give rise to a (modified) Feynman diagram and, therefore, thermal quantities can also have a diagrammatic representation.

In fact, the discussion of diagrams at finite temperature is quite parallel to that at zero temperature. In particular, we note that Wick's theorem can be naturally generalized to finite temperature in this formalism. Furthermore, if we define (for $\tau_1 > \tau_2$)

$$S(\tau_1, \tau_2) = P_\tau \left(\mathrm{e}^{-\int_{\tau_2}^{\tau_1} d\tau H_I'(\tau)} \right) = S(\tau_1) S^{-1}(\tau_2) \tag{1.17}$$

then, this operator satisfies the semigroup properties of the zero temperature time evolution operator, namely,

$$\begin{aligned} S(\tau) &= S(\tau,0) \\ S^{-1}(\tau) &= S(0,\tau) \\ S(\tau_1,\tau_2)S(\tau_2,\tau_3) &= S(\tau_1,\tau_3) \quad \text{for} \quad \tau_1 > \tau_2 > \tau_3 \end{aligned} \tag{1.18}$$

Let us also note that the two point Greens functions can be defined in the Heisenberg picture as (Here we are assuming that the field ϕ is complex.)

$$\begin{aligned} \mathcal{G}_\beta(\tau,\tau') &= \langle P_\tau(\phi_H(\tau)\phi_H^\dagger(\tau'))\rangle_\beta \\ &= Z^{-1}(\beta)\,\mathrm{Tr}\,e^{-\beta\mathcal{H}}P_\tau(\phi_H(\tau)\phi_H^\dagger(\tau')) \end{aligned} \tag{1.19}$$

Here ϕ_H can represent a bosonic or a fermionic field. Incidentally, we note that, by definition, (ϕ and $\phi^\dagger$ are operators in the Schrödinger picture)

$$\begin{aligned} \phi_H(\tau) &= e^{\tau\mathcal{H}}\phi e^{-\tau\mathcal{H}} \\ \phi_H^\dagger(\tau') &= e^{\tau'\mathcal{H}}\phi^\dagger e^{-\tau'\mathcal{H}} \neq (\phi_H(\tau'))^\dagger \end{aligned} \tag{1.20}$$

as we had discussed earlier. We have suppressed all dependence of the fields on spatial coordinates as well as any spinorial indices which are not relevant for our discussion. The τ ordering, like the time ordering at zero temperature, is sensitive to the Grassmann parity of the field variables and is defined as

$$\begin{aligned} &P_\tau(\phi_H(\tau)\phi_H^\dagger(\tau')) \\ &\quad = \theta(\tau-\tau')\phi_H(\tau)\phi_H^\dagger(\tau') \pm \theta(\tau'-\tau)\phi_H^\dagger(\tau')\phi_H(\tau) \end{aligned} \tag{1.21}$$

where the minus sign in the second term is for fermionic fields. The relation between the interaction picture and the Heisenberg picture, in

the present context, takes the form (see eqs. (1.5), (1.9) and (1.15))

$$\begin{aligned} A_H(\tau) &= e^{\tau\mathcal{H}} A\, e^{-\tau\mathcal{H}} \\ &= e^{\tau\mathcal{H}} e^{-\tau\mathcal{H}_0} A_I(\tau) e^{\tau\mathcal{H}_0} e^{-\tau\mathcal{H}} \\ &= S^{-1}(\tau) A_I(\tau) S(\tau) \end{aligned} \tag{1.22}$$

Using eqs. (1.21) and (1.22) as well as the properties in eq. (1.18), it is easy to show that the two point Greens function can also be written as $(0 \leq \tau, \tau' \leq \beta)$

$$\begin{aligned} \mathcal{G}_\beta(\tau,\tau') &= \frac{\mathrm{Tr}\, e^{-\beta\mathcal{H}} P_\tau(S^{-1}(\tau)\phi_I(\tau)S(\tau)S^{-1}(\tau')\phi_I^\dagger(\tau')S(\tau'))}{\mathrm{Tr}\, e^{-\beta\mathcal{H}}} \\ &= \frac{\mathrm{Tr}\, e^{-\beta\mathcal{H}_0} S(\beta) P_\tau(S^{-1}(\tau)\phi_I(\tau)S(\tau)S^{-1}(\tau')\phi_I^\dagger(\tau')S(\tau'))}{\mathrm{Tr}\, e^{-\beta\mathcal{H}_0} S(\beta)} \\ &= \frac{\mathrm{Tr}\, e^{-\beta\mathcal{H}_0} P_\tau(S(\beta)S^{-1}(\tau)\phi_I(\tau)S(\tau)S^{-1}(\tau')\phi_I^\dagger(\tau')S(\tau'))}{\mathrm{Tr}\, e^{-\beta\mathcal{H}_0} S(\beta)} \\ &= \frac{\mathrm{Tr}\, e^{-\beta\mathcal{H}_0} P_\tau(\phi_I(\tau)\phi_I^\dagger(\tau')S(\beta))}{\mathrm{Tr}\, e^{-\beta\mathcal{H}_0} S(\beta)} \\ &= \frac{\langle P_\tau(\phi_I(\tau)\phi_I^\dagger(\tau')S(\beta))\rangle_{\beta,0}}{\langle S(\beta)\rangle_{\beta,0}} \end{aligned} \tag{1.23}$$

where we have used the fact that, since β has the largest value, $S(\beta)$ can be taken inside the τ-ordering, P_τ, and, furthermore, the order of the factors does not matter inside P_τ. Here the subscript "0" represents the fact that the averages are calculated in a noninteracting ensemble. This is exactly like the zero temperature results.

In addition to the similarities, we should also note the differences from the zero temperature case. The most significant, from our point of view, is the fact that unlike at zero temperature, in the present case, the "time" variable is integrated over a finite interval. As a result, as we go to higher and higher orders of expansion, the number of such integration regions grows rapidly and it is not even clear whether the diagrammatic evaluations can even be carried out in a manner which would be useful to carry out calculations.

1.3 Matsubara Frequencies

To tackle the problem of evaluating finite temperature diagrams, let us note that even at zero temperature, the calculation of Feynman diagrams is much easier in the momentum space. Correspondingly, we can ask whether going over to the momentum space will help in the evaluation of diagrams at finite temperature. Let us, however, note first some of the essential properties of the two point Greens functions. From the definition in eq. (1.19) as well as from the relation between the Heisenberg and the Schrödinger pictures, it is straightforward to show that the two point Greens functions depend only on the difference $\tau - \tau'$. Second, each of the time variables can lie between $0 \leq \tau, \tau' \leq \beta$. Consequently, the argument of the two point function has the range $-\beta \leq \tau - \tau' \leq \beta$. Furthermore, from the cyclicity properties of the trace as well as from the definitions of time ordering in eq. (1.21), it is easy to show that

$$\mathcal{G}_\beta(\tau < 0) = \pm \mathcal{G}_\beta(\tau + \beta) \tag{1.24}$$

Or more explicitly, we can write from eqs. (1.19)-(1.21), that for $\tau > 0$,

$$\begin{aligned} \mathcal{G}_\beta(0,\tau) &= \pm\langle\phi_H^\dagger(\tau)\phi_H(0)\rangle_\beta \\ &= \pm\langle\phi_H(\beta)\phi_H^\dagger(\tau)\rangle_\beta \\ &= \pm\mathcal{G}_\beta(\beta,\tau) \end{aligned} \tag{1.25}$$

which is equivalent to eq. (1.24). This result can also be seen to follow from the general KMS relation in eq. (1.6) or (1.7) with appropriate identification (namely, rotation to imaginary times). It simply relates the values of the two point function with negative arguments to those with positive arguments.

We note next that since the Greens functions are defined within a finite time interval, the corresponding Fourier transformation can involve only discrete frequencies. Thus, we can write, on general grounds,

$$\begin{aligned} \mathcal{G}_\beta(\tau) &= \frac{1}{\beta}\sum_n e^{-i\omega_n\tau}\mathcal{G}_\beta(\omega_n) \\ \mathcal{G}_\beta(\omega_n) &= \frac{1}{2}\int_{-\beta}^{\beta} d\tau\, e^{i\omega_n\tau}\mathcal{G}_\beta(\tau) \end{aligned} \tag{1.26}$$

where $\omega_n = \frac{n\pi}{\beta}$ with $n = 0, \pm1, \pm2\ldots$ Even though all integer modes are allowed in the Fourier expansion in eq. (1.26), because of the (anti) periodicity conditions satisfied by $\mathcal{G}_\beta$ (see eqs. (1.24) and (1.25)), only even integer modes contribute to the bosonic Greens function while only the odd integer modes contribute to the fermionic Greens functions.

This is easily seen as follows. Let us consider the second relation in eq. (1.26). Using eq. (1.24), we can also write this as

$$\begin{aligned}
\mathcal{G}_\beta(\omega_n) &= \frac{1}{2}\int_{-\beta}^{0} d\tau\, \mathrm{e}^{i\omega_n\tau}\mathcal{G}_\beta(\tau) + \frac{1}{2}\int_{0}^{\beta} d\tau\, \mathrm{e}^{i\omega_n\tau}\mathcal{G}_\beta(\tau) \\
&= \pm\frac{1}{2}\int_{-\beta}^{0} d\tau\, \mathrm{e}^{i\omega_n\tau}\mathcal{G}_\beta(\tau+\beta) + \frac{1}{2}\int_{0}^{\beta} d\tau\, \mathrm{e}^{i\omega_n\tau}\mathcal{G}_\beta(\tau) \\
&= \pm\frac{1}{2}\int_{0}^{\beta} d\tau\, \mathrm{e}^{i\omega_n(\tau-\beta)}\mathcal{G}_\beta(\tau) + \frac{1}{2}\int_{0}^{\beta} d\tau\, \mathrm{e}^{i\omega_n\tau}\mathcal{G}_\beta(\tau) \\
&= \frac{1}{2}(1 \pm \mathrm{e}^{-i\omega_n\beta})\int_{0}^{\beta} d\tau\, \mathrm{e}^{i\omega_n\tau}\mathcal{G}_\beta(\tau) \\
&= \frac{1}{2}(1 \pm (-1)^n)\int_{0}^{\beta} d\tau\, e^{i\omega_n\tau}\mathcal{G}_\beta(\tau) \qquad (1.27)
\end{aligned}$$

This immediately shows that, for bosons, $\mathcal{G}_\beta(\omega_n)$ vanishes for odd n, while for fermions it vanishes when n is even. Thus we conclude, from eq. (1.27), that the Fourier transformation of the time variable, in the case of the two point function, gives

$$\begin{aligned}
\mathcal{G}_\beta(\tau) &= \frac{1}{\beta}\sum_n e^{-i\omega_n\tau}\,\mathcal{G}_\beta(\omega_n) \\
\mathcal{G}_\beta(\omega_n) &= \int_{0}^{\beta} d\tau\, \mathrm{e}^{i\omega_n\tau}\mathcal{G}_\beta(\tau) \qquad (1.28)
\end{aligned}$$

where

$$\omega_n = \begin{cases} \frac{2n\pi}{\beta} & \text{for bosons} \\ \frac{(2n+1)\pi}{\beta} & \text{for fermions} \end{cases} \qquad (1.29)$$

These are commonly referred to as the Matsubara frequencies.

The spatial coordinates, on the other hand, are continuous just as in the case of zero temperature field theory and, therefore, there is nothing new in their Fourier transformation. Thus, putting in all the coordinates, we can write

$$\begin{aligned} \mathcal{G}_\beta(\vec{x},\tau) &= \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\, e^{-i(\omega_n\tau - \vec{k}\cdot\vec{x})}\, \mathcal{G}_\beta(\vec{k},\omega_n) \\ \mathcal{G}_\beta(\vec{k},\omega_n) &= \int_0^\beta d\tau \int d^3x\, e^{i(\omega_n\tau - \vec{k}\cdot\vec{x})}\, \mathcal{G}_\beta(\vec{x},\tau) \end{aligned} \tag{1.30}$$

where we have assumed that we are in four space-time dimensions (The spatial integrals can, otherwise, be appropriately redefined.) and the allowed frequencies are as defined in eq. (1.29). We are now ready to derive the form of the propagator for any theory. For example, if we take the bosonic Klein-Gordon theory, the zero temperature Greens function satisfies (Our metric in the Minkowski space is (+,−,−,−).)

$$(\partial_\mu\partial^\mu + m^2)G(x) = -\delta^4(x) \tag{1.31}$$

Going over to the τ variable, we note that the finite temperature Greens function would satisfy the equation (When rotating to Euclidean space or imaginary time, we let $t \to -i\tau$, or $p^0 \to ip_4$ which leads to $G \to -\mathcal{G}$.)

$$\left(\frac{\partial^2}{\partial\tau^2} + \nabla^2 - m^2\right)\mathcal{G}_\beta(\vec{x},\tau) = -\,\delta^3(x)\,\delta(\tau) \tag{1.32}$$

Using eq. (1.30) as well as a discrete form of the delta function in the τ variable, then, we conclude that the momentum space Greens function, in this case, becomes

$$\mathcal{G}_\beta(\vec{k},\omega_n) = \frac{1}{\omega_n^2 + \vec{k}^2 + m^2} \tag{1.33}$$

with the appropriate ω_n given in eq. (1.29). The Greens function for the fermions can similarly be obtained in a straightforward manner. The important thing to note from eq. (1.33) is that, unlike the zero temperature case, here the Greens functions do not have singularities for real values of energy and momentum variables.

The analysis of the propagators (Greens functions), in the momentum space, simplifies the diagrammatic calculations at finite temperature considerably. In fact, it makes the finite temperature calculations completely parallel (at least qualitatively) to the zero temperature case. The interaction vertices of the theory are defined as in the zero temperature case. Only the exact forms of the propagators are different from the zero temperature ones and carry the temperature dependence. However, as we will see, this is sufficient to introduce new characters to the quantum mechanical systems at finite temperature.

1.4 Path Integral Formulation

The Matsubara formalism or the imaginary time formalism can also be very easily understood within the path integral context. Let us recall that the transition amplitude in a quantum field theory at zero temperature has the functional representation given by (We will use $\hbar = 1$.)

$$\langle \phi(\vec{x}_1, t_1) | \phi(\vec{x}_2, t_2) \rangle = \langle \phi_1 | e^{-iH(t_1 - t_2)} | \phi_2 \rangle = N' \int \mathcal{D}\phi \, e^{iS} \qquad (1.34)$$

where ϕ is the basic quantum field variable, N' is an irrelevant normalization constant and the action S is defined to be

$$S[\phi] = \int_{t_2}^{t_1} dt \int d^3x \, \mathcal{L} \tag{1.35}$$

with $\mathcal{L}$ representing the appropriate Lagrangian density for the system. The functional integral (path integral), in this case, is defined over paths which satisfy

$$\begin{aligned} \phi(\vec{x}_1, t_1) &= \phi_1 \\ \phi(\vec{x}_2, t_2) &= \phi_2 \end{aligned} \tag{1.36}$$

and the end points are held fixed (not integrated over).

Given eqs. (1.34)-(1.36), it is now clear that if we identify

$$t_1 - t_2 = -i\beta \tag{1.37}$$

then, we can write the partition function for any quantum system as

$$\begin{aligned} Z(\beta) &= \mathrm{Tr}\, e^{-\beta\mathcal{H}} = \int d\phi_1 \, \langle \phi_1 | e^{-\beta\mathcal{H}} | \phi_1 \rangle \\ &= N' \int \mathcal{D}\phi \, e^{-\mathcal{S}_E} \end{aligned} \tag{1.38}$$

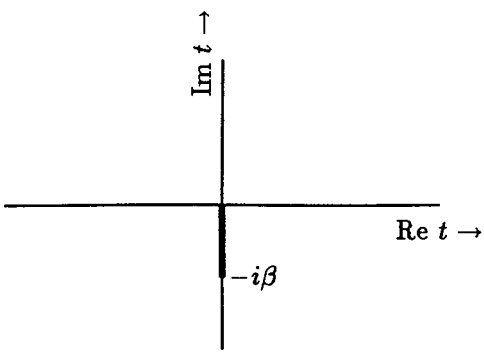
where $\mathcal{S}_E$ is related to the Euclidean (imaginary time) action as (if the chemical potential is nonzero)

$$\begin{aligned} \mathcal{S}_E &= S_E + \beta\mu N \\ &= \int_0^\beta d\tau \int d^3x \, \mathcal{L}_E + \beta\mu N \end{aligned} \tag{1.39}$$

Furthermore, the field variables are assumed to satisfy the (anti) periodicity conditions

$$\phi(\vec{x}, \beta) = \pm\phi(\vec{x}, 0) \tag{1.40}$$

depending on whether the field variables are bosonic or fermionic and the end points, in this case, are integrated over as well. (Periodicity condition for the bosons is obvious from eqs. (1.36) and (1.38). Anti-periodicity for fermions, on the other hand, is not so obvious and we will discuss this in chapter 4, more precisely in section 4.5) As can be seen from eq. (1.34), these conditions essentially give rise to the trace in the definition of the partition function in eq. (1.38). We note here that the contour of integration in the complex time plane, in the present formalism, is along the negative imaginary axis in the following way.



The path integral formulation of the partition function is quite interesting in the sense that it shows how parallel the zero temperature and the finite temperature descriptions of quantum field theories are – at least qualitatively. The definitions of the diagrams

– 1PI, connected etc. – are exactly the same and can be obtained from an expansion of the path integral. The only difference is that the field variables now have to satisfy (anti) periodicity conditions (see eq. (1.40)) and correspondingly should be expanded in an appropriate basis. This leads to the energy values being discrete, eq. (1.29). (This is completely analogous to quantizing a quantum mechanical system in a box – it is, however, a one dimensional box in the τ direction. The topology of space-time, in this description, becomes that of $\mathbf{R}^3 \times S^1$.)

It is also worth noting at this point that the Matsubara (imaginary time) formalism has been developed completely within the context of equilibrium systems. We have given up the time variable in favor of temperature. Thus, this method is ideal for studying static, equilibrium properties of a quantum system. Time dependence can be introduced into Greens functions through a nontrivial analytic continuation which we will discuss later. However, that would only describe slow time developments of quantum mechanical systems in thermal equilibrium. But this formalism is completely unsuitable for studying non-equilibrium phenomenon such as phase transitions. Furthermore, dynamical relations, such as the Ward identities of gauge theories, are hard to study in this formalism. But this is the formalism that is most widely used in the calculation of equilibrium thermodynamic quantities.

To summarize, then, in the Matsubara formalism, the partition function is given a path integral representation where the action for the path integral corresponds to the Euclidean action of the original system with the time integration over a finite interval. Fur-

thermore, the fundamental field variables of this Euclidean action are supposed to satisfy (anti) periodic boundary conditions with a period of β. The Feynman rules for this theory can be read out from the path integral. The vertices are the same as that of the zero temperature Euclidean theory. However, the propagators now correspond to the inverses of the operators of the quadratic part of the Lagrangian defined in a space of (anti) periodic functions. That gives the propagator a nontrivial dependence on the temperature. In this formalism, the propagator does not have a simple decomposition into a zero temperature part and a genuinely temperature dependent part which, as we will see later, will be a property of the real time formulations of finite temperature field theories.

1.5 Applications

As we have seen earlier, in the imaginary time formalism, the only difference between the zero temperature and the finite temperature field theories lies in the form of the propagator which carries all the temperature dependence. The vertices at finite temperature are exactly the same as those at zero temperature. Thus, given any quantum field theory, we can carry out calculations of thermodynamic interest perturbatively by calculating Feynman diagrams. Let us note here that the propagators for the spin 0 and spin 1/2 fields are given respectively by

$$\mathcal{G}_\beta(\vec{k}, \omega_n) = \frac{1}{\omega_n^2 + \vec{k}^2 + m^2}$$

$$
\begin{aligned}
&= \frac{1}{(4n^2\pi^2/\beta^2) + \vec{k}^2 + m^2} \\
S_\beta(\vec{k}, \omega_n) &= \frac{\gamma^0\omega_n + \vec{\gamma}\cdot\vec{k} - m}{\omega_n^2 + \vec{k}^2 + m^2} \\
&= \frac{\gamma^0((2n+1)\pi/\beta) + \vec{\gamma}\cdot\vec{k} - m}{((2n+1)^2\pi^2/\beta^2) + \vec{k}^2 + m^2}
\end{aligned}
\tag{1.41}
$$

With these, let us next calculate the one loop mass correction in a self-interacting bosonic theory.

1.6 One Loop Mass Correction

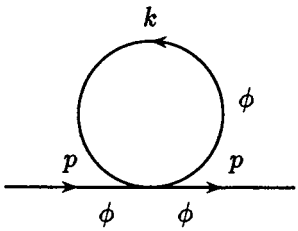
Let us consider a self-interacting ϕ^4–theory described by the Lagrangian density

$$
\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1.42}
$$

According to our discussion so far, if we want to calculate quantities at finite temperature, we should treat time as an imaginary parameter in which case the theory becomes a Euclidean theory. The propagator for the theory can be read out from eq. (1.41) and the vertices are those of the Euclidean theory. The diagrammatic calculations can, then, be carried out analogous to the zero temperature case. The only difference is that since the energy values are now quantized, the intermediate energy integrals have to be replaced by sums over discrete values, namely,

$$
\int \frac{d^4k_E}{(2\pi)^4} \longrightarrow \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \tag{1.43}
$$

The only nontrivial contribution at one loop is that to a mass correction and has the form



$$
\begin{aligned}
-\Delta m^2 &= -\frac{\lambda}{2\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{(4n^2\pi^2/\beta^2)+\omega_k^2} \\
&= -\frac{\lambda}{2\beta}\left(\frac{\beta}{2\pi}\right)^2 \sum_n \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{n^2+(\beta\omega_k/2\pi)^2}
\end{aligned}
\tag{1.44}
$$

where we have introduced the notation

$$
\omega_k = (\vec{k}^2 + m^2)^{1/2} \tag{1.45}
$$

The sum in eq. (1.44) can be easily evaluated using the method of residues which gives

$$
\sum_{n=-\infty}^{\infty} \frac{1}{n^2+y^2} = \frac{\pi}{y}\coth \pi y \qquad \text{for} \quad y>0 \tag{1.46}
$$

Using this, then, we obtain, from eq. (1.44),

$$
\Delta m^2 = \frac{\lambda}{4}\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right) \tag{1.47}
$$

There are several things to note from eq. (1.47). First, from the identity

$$\coth \beta x = 1 + 2n_B(2x)$$

where $n_B(x)$ represents the bosonic distribution function

$$n_B(x) = \frac{1}{e^{\beta x} - 1}$$

it is clear that the mass correction is a sum of two terms – one corresponding to zero temperature and the other explicitly temperature dependent.

$$\begin{aligned} \Delta m^2 &= \Delta m_0^2 + \Delta m_T^2 \\ &= \frac{\lambda}{4} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} + \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} \frac{1}{e^{\beta\omega_k} - 1} \end{aligned} \tag{1.48}$$

Second, the divergence in the expression for the mass correction is entirely contained in the zero temperature part. The temperature dependent part is free from ultraviolet divergences. Therefore, the zero temperature counter terms are sufficient to renormalize the theory. We will come to a further discussion of this point in the next chapter. Furthermore, unlike the zero temperature case, the integral in eq. (1.47), even for this simple process, cannot be evaluated in a closed form. However, with a bit of analysis, it can be shown that at high temperature, the temperature dependent part of the mass correction has the form

$$\Delta m_T^2 = \frac{\lambda}{24\beta^2} + O(\beta m) = \frac{\lambda T^2}{24} + O\left(\frac{m}{T}\right) \tag{1.49}$$

Thus, we see that temperature induces a mass for the bosons analogous to a particle moving in a medium. Furthermore, this mass is positive which is at the heart of studies in the restoration of spontaneously broken symmetries – a topic we will discuss later.

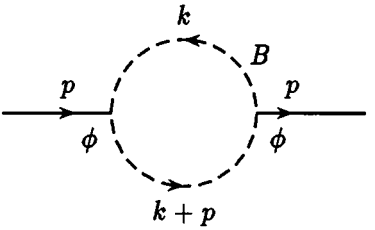
1.7 Self-Energy at Finite Temperature

Let us next study the self-energy in a bosonic theory to bring out some specific features that arise at finite temperature. We can take, as a toy model, an interacting theory of two spin zero bosons described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{M^2}{2}B^2 - \frac{g}{2}\phi B^2 \qquad (1.50)$$

We can think of B as describing a heavy mass particle. We can integrate this field out since the Lagrangian density depends on it only quadratically and ask for its contribution to the self-energy of the lighter mass field. Let us note that the Lagrangian density in eq. (1.50) is not a realistic model in any sense, and should only be treated as a toy model to bring out some particular features at finite temperature.

The propagator for the B-field can be read out from eq. (1.41). The vertices are the same as the zero temperature theory. However, since the energy values are now quantized, as we have discussed earlier, we have to replace the energy integrals in the intermediate vertices by sums (see eq. (1.43)). With these rules, the self-energy for the ϕ–field at one loop becomes (The internal loop consists of the B-propagator and the symmetry factor is $\frac{1}{2}$.)



$$\Pi(\vec{p},p^0) = \frac{g^2}{2\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\frac{1}{\frac{4n^2\pi^2}{\beta^2}+\vec{k}^2+M^2} \times \frac{1}{(\frac{2n\pi}{\beta}+p^0)^2+(\vec{k}+\vec{p})^2+M^2} \tag{1.51}$$

where we have denoted the energy of the external lines as p^0 which is assumed to be discrete at finite temperature in the imaginary time formalism.

Let us next introduce the notations

$$\begin{aligned}\omega_k &= (\vec{k}^2+M^2)^{1/2} \\ \omega_{k+p} &= ((\vec{k}+\vec{p})^2+M^2)^{1/2}\end{aligned} \tag{1.52}$$

Using this, the self-energy can be rewritten as

$$\begin{aligned}\Pi(\vec{p},p^0) &= -(\frac{\beta}{2\pi})^2(\frac{g^2}{2\beta}) \\ &\times \sum_n \int \frac{d^3k}{(2\pi)^3}\frac{1}{4\omega_k\omega_{k+p}}\left(\frac{1}{n+i\frac{\beta\omega_k}{2\pi}}-\frac{1}{n-i\frac{\beta\omega_k}{2\pi}}\right) \\ &\times \left(\frac{1}{n+\frac{\beta p^0}{2\pi}+i\frac{\beta\omega_{k+p}}{2\pi}}-\frac{1}{n+\frac{\beta p^0}{2\pi}-i\frac{\beta\omega_{k+p}}{2\pi}}\right)\end{aligned} \tag{1.53}$$

The sum over n in eq. (1.53) can be simply evaluated using the formula

$$\sum_n \frac{1}{n+ix}\frac{1}{n+iy} = \frac{\pi}{x-y}\left(\coth(\pi x) - \coth(\pi y)\right) \tag{1.54}$$

which leads to

$$\Pi(\vec{p},p^0) = \frac{g^2}{8}\int \frac{d^3k}{(2\pi)^3}\left[\frac{1}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right)\frac{1}{\omega_{k+p}^2-(\omega_k-ip^0)^2}\right.$$
$$\left. + \frac{1}{\omega_{k+p}}\coth\left(\frac{\beta(\omega_{k+p}-ip^0)}{2}\right)\frac{1}{\omega_k^2-(\omega_{k+p}-ip^0)^2} + (p^0 \leftrightarrow -p^0)\right] \tag{1.55}$$

We can now use the fact that the energy for the external lines takes the discrete value, $p^0 = \frac{2l\pi}{\beta}$. Using the periodicity of "coth" for such values of p^0, we obtain

$$\begin{aligned}\Pi(\vec{p},p^0) &= \frac{g^2}{8}\int \frac{d^3k}{(2\pi)^3}\left[\frac{1}{\omega_k}\coth\left(\frac{\beta\omega_k}{2}\right)\frac{1}{\omega_{k+p}^2-(\omega_k-ip^0)^2}\right.\\ &+ \left.\frac{1}{\omega_{k+p}}\coth\left(\frac{\beta\omega_{k+p}}{2}\right)\frac{1}{\omega_k^2-(\omega_{k+p}-ip^0)^2} + (p^0 \leftrightarrow -p^0)\right]\end{aligned} \tag{1.56}$$

This simple calculation already brings out the subtleties associated with finite temperature quantum field theories. First, we note that the one loop integral in eq. (1.56) cannot be evaluated in a closed form. We can only evaluate this quantity in the limits of high and low temperatures. But even so, it is not hard to see that the self-energy at finite temperatures is a non-analytic function of the momentum and

energy at the origin. To see this, let us pretend that p^0 is a continuous variable and analytically continue Π to continuous values of p^0 which would allow us to take the limit $p^0 \to 0$. The analytic continuation of a function to continuous values of its argument is, in general, not well defined if we only know the values of the function at a set of discrete points. However, if the functions are well behaved at infinity (convergent), then this can be done. With this then, we now see that

$$\begin{aligned}\Pi(0,p^0) &= \frac{g^2}{8}\int \frac{d^3k}{(2\pi)^3}\frac{4}{\omega_k}\frac{1}{4\omega_k^2-(p^0)^2}\coth\left(\frac{\beta\omega_k}{2}\right) \\ &\xrightarrow{p^0\to 0} \frac{g^2}{8}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k^3}\coth\left(\frac{\beta\omega_k}{2}\right) \end{aligned} \tag{1.57}$$

On the other hand, the reverse limit will give

$$\begin{aligned}\Pi(\vec{p},0) &= -\frac{g^2}{8}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k\omega_{k+p}} \\ &\quad \left[\frac{1}{\omega_k-\omega_{k+p}}\left(\coth\left(\frac{\beta\omega_k}{2}\right)-\coth\left(\frac{\beta\omega_{\vec{k}+\vec{p}}}{2}\right)\right)\right. \\ &\quad \left.-\frac{1}{\omega_k+\omega_{k+p}}\left(\coth\left(\frac{\beta\omega_k}{2}\right)+\coth\left(\frac{\beta\omega_{\vec{k}+\vec{p}}}{2}\right)\right)\right] \\ &\xrightarrow{\vec{p}\to 0} \frac{g^2}{8}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k^3}\left(\coth\left(\frac{\beta\omega_k}{2}\right)+\frac{\beta\omega_k}{2}\mathrm{csch}^2\left(\frac{\beta\omega_k}{2}\right)\right)\end{aligned} \tag{1.58}$$

This shows that the limits $p^0 \to 0$, $\vec{p} \to 0$ and $\vec{p} \to 0$, $p^0 \to 0$ do not commute and, in fact, the self-energy is non-analytic at the origin in the momentum space.

Such a non-analyticity is unexpected from our experience with zero temperature field theory. That is because Lorentz invariance forces the zero temperature amplitudes to be functions of the external momentum, p^μ, and that makes them analytic. At finite temperature, however, Lorentz invariance is broken by the choice of a specific frame and, therefore, the amplitudes can depend independently on p^0 and $\vec{p}$. The limits $p^0 \to 0, \quad \vec{p} \to 0$ and $\vec{p} \to 0, \quad p^0 \to 0$, therefore, need not be the same. The non-commuting nature of limits is not merely of academic interest – rather, they represent physical effects. In fact, the limit $\vec{p} \to 0, \quad p^0 = 0$ corresponds to taking the static limit and leads to the dynamical screening mass for the electric fields. The other limit, namely, $p^0 \to 0, \quad \vec{p} = 0$, on the other hand, gives the plasmon mass associated with the damping of the oscillations in a plasma and the two masses do not have to coincide. That is physically the origin of the non-analyticity of the self-energy at the origin in the momentum space at finite temperature.

An alternate way to look at the self-energy at finite temperature is as follows. If we introduce the number densities (Here we are dealing with bosons only and, therefore, we will ignore the subscript representing a bosonic distribution for simplicity.)

$$\begin{aligned} n(\omega_k) &= \frac{1}{e^{\beta\omega_k} - 1} \\ n(\omega_{k+p}) &= \frac{1}{e^{\beta\omega_{k+p}} - 1} \end{aligned} \tag{1.59}$$

and note, as pointed out earlier, that

$$\coth\left(\frac{\beta\omega_k}{2}\right) = 1 + 2n(\omega_k) \tag{1.60}$$

then, we can rewrite the expression for the self-energy at finite temperature in eq. (1.56) also as (We denote $\omega = ip^0$ and suppress the argument $\vec{p}$ in Π for simplicity.)

$$\Pi(\omega) = \frac{g^2}{8}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k\omega_{k+p}}$$
$$\Bigg[((1+n(\omega_k))(1+n(\omega_{k+p})) - n(\omega_k)n(\omega_{k+p}))$$
$$\times\left(\frac{1}{\omega+\omega_k+\omega_{k+p}} - \frac{1}{\omega-\omega_k-\omega_{k+p}}\right)$$
$$+\Big(n(\omega_k)(1+n(\omega_{k+p})) - n(\omega_{\vec{k}+\vec{p}})(1+n(\omega_k))\Big)$$
$$\times\left(\frac{1}{\omega-\omega_k+\omega_{k+p}} - \frac{1}{\omega+\omega_k-\omega_{k+p}}\right)\Bigg] \qquad (1.61)$$

Although this function is defined for only imaginary discrete values of ω, we can extend it to the full complex plane satisfying

$$\Pi^*(\omega) = \Pi(\omega^*) \qquad (1.62)$$

With this, it is easy to see that the analytic extension has cuts along the real axis and that the discontinuity across these cuts is pure imaginary for real ω and is given by

$$\text{Disc}\,\Pi(\omega) = \lim_{\eta\to 0}\left(\Pi(\omega+i\eta) - \Pi(\omega-i\eta)\right) = 2i\,\text{Im}\,\Pi(\omega) \qquad (1.63)$$

The discontinuity can be read out directly from eq. (1.61) and is given in terms of

$$\begin{aligned}\mathrm{Im}\,\Pi(\omega) = -\frac{g^2}{16}\int \frac{d^3k}{(2\pi)^2}\,\frac{1}{\omega_k\omega_{k+p}} \\ [\{(1+n(\omega_k))(1+n(\omega_{k+p})) - n(\omega_k)n(\omega_{k+p})\} \\ \times(\delta(\omega+\omega_k+\omega_{k+p}) - \delta(\omega-\omega_k-\omega_{k+p})) \\ +\{n(\omega_k)(1+n(\omega_{k+p})) - n(\omega_{k+p})(1+n(\omega_k))\} \\ \times(\delta(\omega-\omega_k+\omega_{k+p}) - \delta(\omega+\omega_k-\omega_{k+p}))] \qquad (1.64)\end{aligned}$$

This brings out an interesting feature of field theories at finite temperature. We note that, at zero temperature, $\mathrm{Im}\,\Pi$ represents a decay rate which is the square of the decay amplitude integrated over the appropriate phase space factors. At finite temperature, however, the available density of states has to be appropriately weighted by statistical factors. Furthermore, there are real particles present in a medium and consequently, there are channels available at finite temperature that are not possible at zero temperature. For example, in the present example, we note that the direct process

$$\phi \longrightarrow B + B$$

would, of course, occur with a statistical weight factor $(1+n(\omega_k))(1+n(\omega_{k+p}))$ (Think of the stimulated emission in a medium for a system such as the laser.). But more importantly, there are real B's in the thermal medium and, therefore, the inverse process

$$B + B \longrightarrow \phi$$

is also possible. However, this will take place with a statistical weight factor $n(\omega_k)n(\omega_{k+p})$. The total rate for the decay of ϕ, therefore, should

correspond to the difference of these two rates and we note that both these processes are indeed included in the first term in eq. (1.64). The other terms, similarly, represent the processes

$$\phi + B \longrightarrow B$$

$$B \longrightarrow \phi + B$$

It is worth noting here that if the decay products were fermions, as opposed to bosons as in the present example, then the statistical factors would have the form $(1-n)$ corresponding to a Pauli blocking representing the fact that there is a real distribution of fermions in the medium of density n and, therefore, the available density of states for the decay products is suppressed.

In general, the location of the branch cuts can also be read out from eq. (1.64). Let us assume that the masses of the particles in the loop are given by m_1 and m_2 respectively (for generality) so that we can define (see eq. (1.52))

$$\begin{aligned} \omega_1 &= \omega_k = (\vec{k}^2 + m_1^2)^{\frac{1}{2}} \\ \omega_2 &= \omega_{k+p} = ((\vec{k}+\vec{p})^2 + m_2^2)^{\frac{1}{2}} \end{aligned} \tag{1.65}$$

Let us next define a variable

$$s = \omega^2 - \vec{p}^{\,2} \tag{1.66}$$

We note now, using Schwartz inequality, that

$$\begin{aligned} \omega_1\omega_2 &= (|\vec{k}|^2 + m_1^2)^{\frac{1}{2}}(|\vec{k}+\vec{p}|^2 + m_2^2)^{\frac{1}{2}} \\ &\geq |\vec{k}||\vec{k}+\vec{p}| + m_1 m_2 \\ \text{or,} \quad \omega_1\omega_2 - m_1 m_2 &\geq |\vec{k}||\vec{k}+\vec{p}| \end{aligned} \tag{1.67}$$

It is now clear that, if in eq. (1.64), the delta function constraint that is satisfied is $\omega = \omega_1 + \omega_2$, then, using eq. (1.65), we obtain

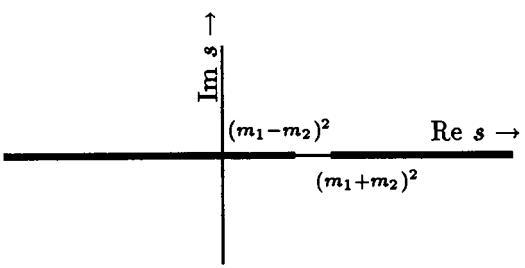
$$\begin{aligned} s &= \omega^2 - \vec{p}^2 = (\omega_1+\omega_2)^2 - \vec{p}^2 \\ &= m_1^2 + m_2^2 + 2\vec{k}\cdot(\vec{k}+\vec{p}) + 2\omega_1\omega_2 \\ &= (m_1+m_2)^2 + 2\vec{k}\cdot(\vec{k}+\vec{p}) + 2(\omega_1\omega_2 - m_1 m_2) \\ &\geq (m_1+m_2)^2 + 2(|\vec{k}||\vec{k}+\vec{p}| + \vec{k}\cdot(\vec{k}+\vec{p})) \\ &\geq (m_1+m_2)^2 \end{aligned} \tag{1.68}$$

This gives the first branch cut of the two point function lying between $(m_1+m_2)^2 \leq s \leq \infty$ which coincides with the zero temperature cut that describes the standard threshold for particle decays.

In addition, however, because of the additional processes at finite temperature, a new cut develops. This is easy to see in the following way. Let us suppose that the delta function constraint in eq. (1.64) that is satisfied is $\omega = \omega_1 - \omega_2$ (or the one with $\omega_1 \leftrightarrow \omega_2$). Then, using eqs. (1.65) and (1.67), we now see that

$$\begin{aligned} s &= \omega^2 - \vec{p}^{\,2} = (\omega_1 - \omega_2)^2 - \vec{p}^{\,2} \\ &= m_1^2 + m_2^2 + 2\vec{k}\cdot(\vec{k}+\vec{p}) - 2\omega_1\omega_2 \\ &= (m_1 - m_2)^2 + 2\vec{k}\cdot(\vec{k}+\vec{p}) - 2(\omega_1\omega_2 - m_1 m_2) \\ &\leq (m_1 - m_2)^2 - 2(|\vec{k}||\vec{k}+\vec{p}| - \vec{k}\cdot(\vec{k}+\vec{p})) \\ &\leq (m_1 - m_2)^2 \end{aligned} \tag{1.69}$$

This shows that, at finite temperature, there is an additional branch cut along $-\infty \leq s \leq (m_1 - m_2)^2$ where the lower limit is obtained for space like s. This new cut corresponds to particle absorption from the medium and pictorially, the cut structure, at finite temperature, is as follows



We note that the first branch cut, of course, does not lead to any non-analyticity at the origin. However, the second branch cut is the one responsible for the non-commuting limits when $m_1 = m_2$ which we have discussed earlier. This analysis also makes it clear that for processes where the loop would involve particles with distinct masses, the non-

analyticity would be absent. This can, in fact, be explicitly checked by calculating the self-energy of the gauge boson in a spontaneously broken gauge theory in the unitary gauge. It is also worth noting here that the non-analyticity is not a special feature of the self-energy of a boson. It is present even for a fermion. In fact, this non-analyticity manifests itself in the calculation of effective actions as well. In particular, one has to define what one means by an effective potential when the effective action is non-analytic at the origin in the momentum space.

1.8 Covariant Description

Naively, one thinks that the very definition of a system at finite temperature breaks Lorentz invariance. This is because, the rest frame of the heat bath already selects out a specific Lorentz frame. This is, however, not true and statistical mechanics can, in fact, be formulated in a manifestly Lorentz covariant fashion. This is particularly useful if we want to take advantage of concepts such as tensor decomposition of amplitudes.

Let us consider a heat bath (or a fluid) with a proper four velocity u^μ satisfying

$$u^2 = u_\mu u^\mu = 1 \tag{1.70}$$

In the rest frame of the heat bath, the four velocity has the form $u^\mu = (1, 0, 0, 0)$ and the inverse temperature, β, is defined in this rest frame.

Given u^μ, we can decompose any other four vector into parallel and orthogonal components with respect to the proper velocity. Thus, for example, an arbitrary four vector, A^μ, will have components parallel and orthogonal to u^μ given by

$$\begin{aligned} A^\mu_{||} &= (A \cdot u)u^\mu \\ A^\mu_{\perp} &\equiv \tilde{A}^\mu = A^\mu - (A \cdot u)u^\mu \end{aligned} \tag{1.71}$$

In particular, we note that if p^μ represents the four momentum of a particle, then its components parallel and perpendicular to the four velocity will be given by

$$\begin{aligned} p^\mu_{||} &= (p \cdot u)u^\mu \equiv \omega u^\mu \\ \tilde{p}^\mu &= p^\mu - \omega u^\mu \end{aligned} \tag{1.72}$$

where we have defined

$$\omega = (p \cdot u) \tag{1.73}$$

It is easy to see from the definitions in eq. (1.72) that in the rest frame of the heat bath

$$\begin{aligned} \omega &= p^0 = p_0 \\ \tilde{p}^\mu &= (0, \vec{p}) \end{aligned} \tag{1.74}$$

It can also be easily checked now using eqs. (1.70), (1.72) and (1.74) that

$$\tilde{p}^2 = \tilde{p}^\mu \tilde{p}_\mu = p^2 - \omega^2 \equiv -k^2 < 0 \tag{1.75}$$

Thus, we can think of ω as the Lorentz invariant energy and $\tilde{p}$ as the Lorentz invariant spatial momentum. Clearly, then, we can write, from eqs. (1.70)-(1.75),

$$p^2 = (\omega u^\mu + \tilde{p}^\mu)(\omega u_\mu + \tilde{p}_\mu) = \omega^2 - k^2 \tag{1.76}$$

and this shows that we can think of k as the length of the momentum (three) vector.

The bosonic and the fermionic distribution functions, now, can be generalized to an arbitrary frame simply as

$$n = \frac{1}{e^{\beta|\omega|} \mp 1} = \frac{1}{e^{\beta|u\cdot p|} \mp 1} \tag{1.77}$$

where $\mp$ stands respectively for bosonic and fermionic distributions. We can also carry out Lorentz tensor decomposition of various amplitudes in an arbitrary frame at finite temperature. Thus, for example, let us note that the projection operator onto the space of vectors which are transverse to u^μ is given by

$$\tilde{\eta}^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu \tag{1.78}$$

Just as we can decompose any vector into components parallel and perpendicular to u^μ, we can, similarly, decompose any vector into components which are parallel and perpendicular to a given momentum vector, say p^μ. Thus, let A^μ represent an arbitrary four vector. Then, its component parallel to p^μ is, of course, given by $\frac{1}{p^2}(p \cdot A)p^\mu$. And the component perpendicular to p^μ has the form

$$\overline{A}^{\mu} = A^{\mu} - \frac{1}{p^2}(p \cdot A)p^{\mu} \tag{1.79}$$

Similarly, the orthogonal components of u^{μ} and $\eta^{\mu\nu}$ with respect to p^{μ} are respectively given by

$$\begin{aligned} \overline{u}^{\mu} &= u^{\mu} - \frac{\omega}{p^2}p^{\mu} \\ \overline{\eta}^{\mu\nu} &= \eta^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2} \end{aligned} \tag{1.80}$$

Given these, it is easy to check that there are only three possible second rank, symmetric tensors that one can construct at finite temperature from p^{μ}, u^{μ} and $\eta^{\mu\nu}$ which are orthogonal to p^{μ}, namely,

$$\begin{aligned} A^{\mu\nu} &= \widetilde{\eta}^{\mu\nu} - \frac{\widetilde{p}^{\mu}\widetilde{p}^{\nu}}{\widetilde{p}^2} \\ B^{\mu\nu} &= \frac{p^2}{\widetilde{p}^2}\,\overline{u}^{\mu}\overline{u}^{\nu} \\ C^{\mu\nu} &= \eta^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2} \end{aligned} \tag{1.81}$$

Each of the structures is normalized so that they act like projection operators. But they are not independent. In fact, it is easy to verify that

$$C^{\mu\nu} = A^{\mu\nu} + B^{\mu\nu} \tag{1.82}$$

Normally, one takes $A^{\mu\nu}$ and $B^{\mu\nu}$ as independent since they are also orthogonal to each other. This, then, allows us to write down the decomposition of the transverse part of the self-energy of a vector field at finite temperature as

$$\begin{aligned}\Pi^{(\beta)}_{\mu\nu}(p) &= A_{\mu\nu}\Pi^{(\beta)}_T(p) + B_{\mu\nu}\Pi^{(\beta)}_L(p) \\ &= \left(\tilde{\eta}_{\mu\nu} - \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2}\right)\Pi^{(\beta)}_T(p) + \frac{p^2}{\tilde{p}^2}\overline{u}_\mu \overline{u}_\nu \Pi^{(\beta)}_L(p) \qquad (1.83)\end{aligned}$$

At zero temperature, we know that the transverse polarization tensor has the form

$$\Pi_{\mu\nu}(p) = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Pi(p)$$

so that, at zero temperature, we can identify (which follows from eqs. (1.81)-(1.82))

$$\Pi_T(p) = \Pi_L(p)$$

It is easy to check from eq. (1.83), on the other hand, that at finite temperature

$$\begin{aligned}\Pi^{(\beta)}_L(p) &= \frac{p^2}{\tilde{p}^2}u^\mu u^\nu \Pi^{(\beta)}_{\mu\nu} \\ (D-2)\Pi^{(\beta)}_T(p) &= \eta^{\mu\nu}\Pi^{(\beta)}_{\mu\nu} - \Pi^{(\beta)}_L \qquad (1.84)\end{aligned}$$

Here D denotes the number of space-time dimensions and it is immediately clear that eq. (1.84) does not give any information on the transverse part of the self-energy when $D = 2$. Such decomposition of tensors at finite temperature can be extended to other amplitudes as well and is often quite useful.

1.9 References

Abrikosov, A.A, L.P. Gorkov and **I.E. Dzyaloshinski**, "Methods of Quantum Field Theory in Statistical Physics", Dover Publication (1975).

Abrikosov, A.A, L.P. Gorkov and **I.E. Dzyaloshinski**, Soviet Physics JETP **9** (1959) 636.

Arnold, P, P. Bedaque, A. Das and **S. Vokos**, Phys. Rev. **D47** (1993) 498.

Fetter, A.L. and **J.D. Walecka**, "Quantum Theory of Many Particle Systems", McGraw-Hill (1971).

Gross, D, R. Pisarski and **L. Yaffe**, Rev. Mod. Phys. **53** (1981) 43.

Kubo, R, J. Phys. Soc. Japan, **12** (1957) 570.

Martin, P and **J. Schwinger**, Phys. Rev. **115** (1959) 1342.

Matsubara, T, Progress of Theoretical Physics, **14** (1955) 351.

Umezawa, H, Y. Tomozawa and **H. Ezawa**, Nuovo Cimento **5** (1957) 810.

Weldon, H.A, Phys. Rev. **D26** (1982) 1394; *ibid* **D28** (1983) 2007.

Chapter 2

Real Time Formalism

2.1 Closed Time Path Formalism

Our discussion, so far, has concentrated on the imaginary time formalism applied to statistical systems. As we have seen, this formalism is very convenient and calculations are simple if we are interested in static, equilibrium properties. In fact, by definition, it starts out with an equilibrium distribution of particles where the time variable is traded in for the temperature. Time dependence can, of course, be brought into the imaginary time formalism through analytic continuation, but this is nontrivial. We will discuss the analytic continuation later, but let us note here that this can only describe a slow time

dependence. A system far away from equilibrium cannot be described in the imaginary time formalism.

On the other hand, there are systems where we may be interested in dynamical questions. This may involve systems evolving through a phase transition. Study of the evolution of the universe or the study of chiral symmetry breaking in QCD are just examples of questions of this kind. Or, we may simply be interested in the Ward identities in a gauge theory in equilibrium. In either case, the time variable plays a fundamental role and cannot be traded in for an equilibrium temperature. The real time formalism handles such questions in a natural manner. Here one has a time variable which describes the dynamics of the system and a temperature variable which represents the equilibrium temperature if we are studying a system in thermal equilibrium. While the real time formalism naturally answers dynamical questions (and, of course, also gives static equilibrium properties), it is technically far more complicated than the imaginary time formalism. There are various reasons for this. First, there are subtleties that show up in this formalism which one should be careful about. Second, the real time formalism has not been as widely used as the imaginary time formalism and, therefore, not many things are standardly known about it. But as we will see, the real time formalism provides a much more physical description of what happens at finite temperature.

There are several formulations of the real time formalism and they all lead to equivalent results – at least in the equilibrium case. Here we will describe only one formulation – commonly known as the

closed time path formalism – because we believe that this is potentially richer and unifies both equilibrium and nonequilibrium processes in a natural way. However, other approaches have their own advantages and we will return to another approach – thermofield dynamics – in the next chapter.

To describe the closed time path formalism in a simple manner, let us consider the Schrödinger picture description of a general, quantum mechanical system in a mixed state described by a density matrix ρ. The system is embedded in an external surrounding and may or may not be in thermal equilibrium. In such a case, as we know, the density matrix is given by

$$\rho(t) = \sum_n p_n |\psi_n(t)\rangle\langle\psi_n(t)| \tag{2.1}$$

Here, p_n describes the probability of finding the quantum mechanical system in the state $|\psi_n(t)\rangle$ and, for simplicity, we are assuming that the states of the system are discrete. Being a probability, p_n satisfies

$$\sum_n p_n = 1 \tag{2.2}$$

The average of any operator associated with this system can be calculated from a knowledge of the density matrix and, in the Schrödinger picture takes the form

$$\langle A\rangle(t) = \mathrm{Tr}\,\rho(t)A = \sum_n p_n \langle\psi_n(t)|A|\psi_n(t)\rangle \tag{2.3}$$

It is clear, therefore, that the average of any operator, in such a case, develops a time dependence which is determined from the time dependence of the density matrix. We also note that we can define an entropy associated with the system in a standard way, namely,

$$S = -\langle \ln p \rangle = -\sum_n p_n \ln p_n \tag{2.4}$$

which can be easily seen to be positive semi-definite in view of eq. (2.2).

The time evolution of the density matrix can be obtained from the evolution of the states of the system. If the states of the system satisfy ($\hbar = 1$)

$$i\frac{\partial|\psi(t)\rangle}{\partial t} = H|\psi(t)\rangle \tag{2.5}$$

where H is the Hamiltonian of the system, then, from eq. (2.1), it is clear that the evolution of the density matrix is given by the quantum Liouville equation (which differs from the Heisenberg equation only in an overall sign)

$$i\frac{\partial\rho(t)}{\partial t} = [H, \rho(t)] \tag{2.6}$$

We have assumed here that the probabilities, p_n, do not change with time which amounts to saying that the entropy is a constant in time. The reason for such an assumption is the lack of knowledge, on our part, as to the dynamical evolution of the external system such as the heat bath. However, adiabatic evolutions do arise frequently in our study of physical systems and we will continue with this assumption.

It follows from eq. (2.6) now that if the Hamiltonian, H, governing the dynamics of the quantum system is time independent, then we can write

$$\rho(t) = \mathrm{e}^{-iHt}\,\rho(0)\,\mathrm{e}^{iHt} \tag{2.7}$$

Furthermore, if the Hamiltonian commutes with $\rho(0)$, then the density matrix will be a constant in time and will describe a system in equilibrium. This is true, for example, if the states of the system are stationary states. This is also true if the probabilities have a Boltzmann distribution. The second case is what we will call a thermal equilibrium. But let us note that in general, the system may not be in equilibrium and the Hamiltonian may have a time dependence in which case, we can express the density matrix through the time evolution operator as

$$\rho(t) = U(t,0)\,\rho(0)\,U^{\dagger}(t,0) = U(t,0)\,\rho(0)\,U(0,t) \tag{2.8}$$

where the time evolution operator is defined in the standard manner as

$$\begin{aligned} i\frac{\partial U(t,t')}{\partial t} &= H(t)U(t,t') \\ \text{or,}\quad U(t,t') &= T\left(\mathrm{e}^{-i\int_{t'}^{t} dt''\,H(t'')}\right) \end{aligned} \tag{2.9}$$

with the condition that

$$U(t,t) = 1 \tag{2.10}$$

which also satisfies (Compare these with eqs. (1.13), (1.17) and (1.18))

$$\begin{aligned} U(t_1,t_2)\,U(t_2,t_1) &= 1 \\ U(t_1,t_2)U(t_2,t_3) &= U(t_1,t_3) \qquad \text{for} \qquad t_1 > t_2 > t_3 \end{aligned} \tag{2.11}$$

Since the density matrix, $\rho(0)$, is a positive Hermitian matrix with unit trace, without loss of generality, we can assume that it has the form

$$\rho(0) = \frac{e^{-\beta H_i}}{\mathrm{Tr}\, e^{-\beta H_i}} \tag{2.12}$$

for some Hermitian operator H_i. Let us next suppose that the quantum system is governed by a Hamiltonian of the form

$$H(t) = \begin{cases} H_i & \text{for} \quad \mathrm{Re}\, t \leq 0 \\ \mathcal{H}(t) & \text{for} \quad \mathrm{Re}\, t \geq 0 \end{cases} \tag{2.13}$$

The physical meaning of this is quite clear. For negative times, we prepare the system in a equilibrium state with temperature β and let the system evolve, for positive times, with the true Hamiltonian of the system, $\mathcal{H}(t)$, which may be time dependent. It is clear that if $\mathcal{H}(t) = H_i$, then the system will evolve in equilibrium, but not otherwise.

From eqs. (2.8)-(2.13), it is clear that we can also write

$$\rho(0) = \frac{U(T - i\beta, T)}{\mathrm{Tr}\, U(T - i\beta, T)}$$

and from eq. (2.8) we, then, obtain

$$\begin{aligned} \rho(t) &= U(t,0)\, \rho(0)\, U(0,t) \\ &= \frac{U(t,0) U(T - i\beta, T) U(0,t)}{\mathrm{Tr}\, U(T - i\beta, T)} \end{aligned} \tag{2.14}$$

where T represents a large negative time (for which the "Hamiltonian" of the system, H_i, is time independent). The time evolution of the average of any operator can now be calculated and since this is at the heart of

the closed time path formalism, we will go through the derivation in some detail. We note from eqs. (2.3) and (2.14) that we can write

$$\begin{aligned}
\langle A\rangle(t) &= \mathrm{Tr}\,\rho(t)A \\
&= \frac{\mathrm{Tr}\,U(t,0)U(T-i\beta,T)U(0,t)A}{\mathrm{Tr}\,U(T-i\beta,T)} \\
&= \frac{\mathrm{Tr}\,U(T-i\beta,T)U(0,t)AU(t,0)U(0,T)U(T,0)}{\mathrm{Tr}\,U(T-i\beta,T)} \\
&= \frac{\mathrm{Tr}\,U(T-i\beta,T)U(T,0)U(0,t)AU(t,0)U(0,T)}{\mathrm{Tr}\,U(T-i\beta,T)} \\
&= \frac{\mathrm{Tr}\,U(T-i\beta,T)U(T,t)AU(t,T)}{\mathrm{Tr}\,U(T-i\beta,T)} \qquad (2.15)
\end{aligned}$$

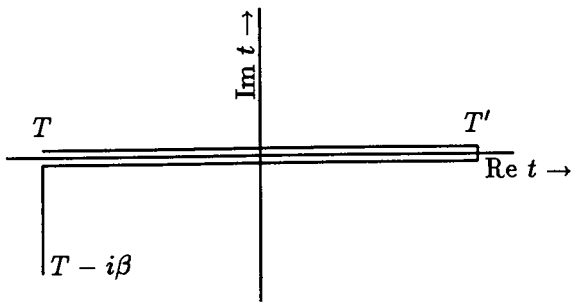
Here we have used the cyclicity of the trace as well as eqs. (2.11) and the fact that for $T<0$, $U(T-i\beta,T)$ and $U(T,0)$ commute.

There is one final step in bringing the expression in eq. (2.15) into the standard form. Let T' represent a large positive time. Then using the property in eq. (2.11), we can write

$$\begin{aligned}
\langle A\rangle(t) &= \frac{\mathrm{Tr}\,U(T-i\beta,T)U(T,T')U(T',T)U(T,t)AU(t,T)}{\mathrm{Tr}\,U(T-i\beta,T)} \\
&= \frac{\mathrm{Tr}\,U(T-i\beta,T)U(T,T')U(T',t)AU(t,T)}{\mathrm{Tr}\,U(T-i\beta,T)U(T,T')U(T',T)} \qquad (2.16)
\end{aligned}$$

The physical meaning of this expression is quite clear. The system evolves from a large negative time T to some time t where an operator insertion of A is introduced. Then, the system evolves from t to a large positive time T' after which it evolves backward in time to T and finally along an imaginary time. Thus, the contour, C, in the complex t-plane

has the form (The paths along the real axis are only infinitesimally displaced along the imaginary axis.)



Eventually, of course, we would like to take the limits $T \rightarrow -\infty$ and $T' \rightarrow \infty$ in which case the form of the average in eq. (2.16) suggests that we should define a generating functional of the form

$$Z[J_c] = \mathrm{Tr}\, U_{J_c}(T - i\beta, T) U_{J_c}(T, T') U_{J_c}(T', T) \tag{2.17}$$

where J_c defines a source which is defined along the time contour that we have described. We should also note that U_{J_c} represents the evolution operator in the presence of the external sources defined along the contour. For a time-independent Hamiltonian and for $J_c = J$, namely, if the source is the same along the entire contour, then it is clear that the generating functional in eq. (2.17) is nothing other than the one that we have discussed for the imaginary time formalism since, in this case,

$$U_{J_c}(T, T') U_{J_c}(T', T) = 1$$

which follows from the properties in eq. (2.11). In such a case, we will have a real time description of a system in equilibrium. On the other hand, the sources need not coincide all along the time path (in particular, along the two branches going forward and backward in time) just as the Hamiltonian can be different for different times. If that is the case, then the generating functional can describe a system not in equilibrium.

2.2 Propagators

It is clear now that if we take eq. (2.17) as defining the generating functional of our theory, then we can give it a path integral representation much like at zero temperature. Thus, for example, for a real Klein-Gordon theory interacting only with an external source, we can write the generating functional as (see eqs. (1.37)-(1.40))

$$Z[J_c] = \int \mathcal{D}\phi \, e^{i \int_c dt \int d^3x \, (\mathcal{L} + J_c \phi)} \tag{2.18}$$

where we can choose the free Lagrangian density to have the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 \tag{2.19}$$

Here the field, ϕ, has to satisfy periodic boundary condition (see eq. (1.40)) and the time integral is defined along the contour we have described earlier which has mainly three parts. We will denote the branch going from $-\infty$ to ∞ above the real axis as C_+, the return path from ∞ to $-\infty$ below the real axis as C_- and the final segment along the imaginary axis as C_3. We will show shortly that the last segment contributes

in a trivial way to the generating functional and hence can be ignored. For all practical purposes, therefore, we can think of the contour as consisting of C_+ and C_- which defines a closed contour in the complex time plane. This is why this formalism is also commonly referred to as the closed time path formalism. We note here that the time integration along this closed contour can also be written, with our convention, as

$$\int_c dt = \int_{-\infty}^{\infty} dt_+ - \int_{-\infty}^{\infty} dt_- \tag{2.20}$$

Given a path integral for the partition function as in eq. (2.18), we can obtain various Greens functions by taking functional derivatives with respect to the sources and these would correspond to appropriate ensemble averages. However, these Greens functions, unlike the zero temperature ones, will be time ordered along the contour, C, in the complex time plane. Let us denote such contour ordered Greens functions by $\langle T_c(A(t_1)B(t_2)\cdots)\rangle$. Thus, for example, for the real Klein-Gordon theory in eqs. (2.18)-(2.19), we can define the propagator as

$$\begin{aligned} iG_c(t-t') &= \langle T_c(\phi(t)\phi(t'))\rangle \\ &= \theta_c(t-t')\langle\phi(t)\phi(t')\rangle + \theta_c(t'-t)\langle\phi(t')\phi(t)\rangle \\ &= (-i)^2 \frac{1}{Z[J_c]} \frac{\delta^2 Z[J_c]}{\delta J_c(t)\delta J_c(t')}\bigg|_{J_c=0} \end{aligned} \tag{2.21}$$

where t and t' are any two points on the contour and where we have suppressed the spatial coordinates for simplicity. The theta function on the contour is defined in a natural way such that

$$\theta_c(t-t') = \begin{cases} \theta(t-t') & t,t' \quad \text{on } C_+ \\ \theta(t'-t) & t,t' \quad \text{on } C_- \\ 0 & t \text{ on } C_+,\ t' \text{ on } C_- \\ 1 & t' \text{ on } C_+,\ t \text{ on } C_- \end{cases} \tag{2.22}$$

and this leads to a definition of the delta function on the contour as

$$\delta_c(t-t') = \frac{d\theta_c(t-t')}{dt} = \begin{cases} \delta(t-t') & t,t' \text{ on } C_+ \\ -\delta(t-t') & t,t' \text{ on } C_- \\ 0 & \text{otherwise} \end{cases} \tag{2.23}$$

which clearly satisfies (see eq. (2.20))

$$\int_c dt'\, \delta_c(t-t')\, f(t') = f(t)$$

The generating functional in eq. (2.18) can be easily evaluated and has the form

$$Z[J_c] = N\, \mathrm{e}^{-\frac{i}{2}\int_c dt dt' \int d^3x d^3y\, J_c(\vec{x},t) G_c(\vec{x}-\vec{y},t-t') J_c(\vec{y},t')} \tag{2.24}$$

where N is an unimportant normalization constant and G_c satisfies the equation (see, for example, eq. (1.31))

$$\left(\partial_\mu \partial^\mu + m^2\right) G_c(\vec{x}-\vec{y},t-t') = -\delta^3(x-y)\delta_c(t-t') \tag{2.25}$$

Taking the Fourier transform of the spatial coordinates, we can rewrite this equation also as

$$\left(\frac{\partial^2}{\partial t^2} + \omega_k^2\right) G_c(t-t',\omega_k) = -\delta_c(t-t') \tag{2.26}$$

where, as before,

$$\omega_k = (\vec{k}^2 + m^2)^{1/2}$$

The solution of this, subject to the periodicity condition (see eq. (1.7)), can be easily determined to be

$$\begin{aligned} G_c(t-t',\omega) &= \frac{n_B(\omega)}{2i\omega}\left[\theta_c(t-t')\left(e^{\beta\omega - i\omega(t-t')} + e^{i\omega(t-t')}\right)\right. \\ &\left. + \theta_c(t'-t)\left(e^{-i\omega(t-t')} + e^{\beta\omega + i\omega(t-t')}\right)\right] \end{aligned} \tag{2.27}$$

where the bosonic distribution function, n_B, is defined as in eq. (1.59). There are several things to note from the expression for the propagator in eq. (2.27). First, in the zero temperature limit, $\beta \to \infty$, the propagator reduces to the conventional one on C_+. Second, the propagator is an even function very much like at zero temperature, namely,

$$G_c(t-t') = G_c(t'-t)$$

It satisfies the KMS condition (see eq. (1.7)). Finally, let us note that if t is on the real branch, C_+ (or C_-), of the time contour and t' on C_3 (or vice versa), then the propagator vanishes, namely,

$$\begin{aligned} &\lim_{T\to-\infty} G_c(t-T+is,\omega) \\ &= \lim_{T\to-\infty}\left(e^{-i\omega(t-T+is)} + e^{\beta\omega+i\omega(t-T+is)}\right) \to 0 \end{aligned} \tag{2.28}$$

This shows that the propagator cannot connect two distinct points when one of them lies on C_+ (or C_-) and the other on C_3. In other words, the generating functional in eq. (2.24) cannot have mixed terms with J_3

and J_+ (J_-). Therefore, the contribution coming from the branch C_3, to the path integral, factorizes and, then, we can safely set the sources on the branch C_3 to zero. This contribution can be absorbed into a temperature dependent normalization factor which is unimportant as far as dynamical questions are concerned and we can ignore this branch of the contour from now on. Consequently, the relevant contour, in this formalism, really corresponds to a closed path in the complex t-plane. We also note here without going into details that the propagator for the fermionic fields can be similarly derived in a straightforward manner.

2.3 Matrix Structure for Propagators

It is clear from our discussion of the earlier sections that the propagator for any theory can be derived in a straightforward manner very much like the zero temperature case, once we impose the proper periodicity conditions. It is also clear from eqs. (2.23) and (2.27) that at nonzero temperatures, the propagator has more structure than is present at zero temperature. Thus, for example, on the two branches C_+ and C_- of the closed time path, there are four nontrivial propagator structures possible and are obtained from (Propagators are simply "i" times the Greens functions, $G_{\pm\pm}$.)

$$\begin{aligned} G_{++}(t-t',\omega) &= -\frac{i}{2\omega}\Big((\theta(t-t')+n_B(\omega))\,\mathrm{e}^{-i\omega(t-t')} \\ &\qquad +(\theta(t'-t)+n_B(\omega))\,\mathrm{e}^{i\omega(t-t')}\Big) \\ G_{+-}(t-t',\omega) &= -\frac{i}{2\omega}\Big(n_B(\omega)\,\mathrm{e}^{-i\omega(t-t')}+(1+n_B(\omega))\,\mathrm{e}^{i\omega(t-t')}\Big) \end{aligned}$$

$$
\begin{aligned}
G_{-+}(t-t',\omega) &= -\frac{i}{2\omega}\left((1+n_B(\omega))\,\mathrm{e}^{-i\omega(t-t')} + n_B(\omega)\,\mathrm{e}^{i\omega(t-t')}\right) \\
G_{--}(t-t',\omega) &= -\frac{i}{2\omega}\left((\theta(t'-t)+n_B(\omega))\,\mathrm{e}^{-i\omega(t-t')}\right. \\
&\qquad \left. +(\theta(t-t')+n_B(\omega))\,\mathrm{e}^{i\omega(t-t')}\right) \qquad (2.29)
\end{aligned}
$$

Here the subscripts for the Greens functions denote the branches of the contour in the complex time plane on which the appropriate time coordinates reside. In terms of ensemble averages, these have the definitions, from eqs. (2.21) and (2.22),

$$
\begin{aligned}
iG_{++}(t-t') &= \langle T(\phi(t)\phi(t'))\rangle \quad t,t' \in C_+ \\
iG_{+-}(t-t') &= \langle \phi(t')\phi(t)\rangle \quad t \in C_+,\ t' \in C_- \\
iG_{-+}(t-t') &= \langle \phi(t)\phi(t')\rangle \quad t \in C_-,\ t' \in C_+ \\
iG_{--}(t-t') &= \langle T^*(\phi(t)\phi(t'))\rangle \quad t,t' \in C_-
\end{aligned} \qquad (2.30)
$$

where T^* denotes anti-time ordering.

This form of the propagator suggests that we can define a doublet of field variables and sources as

$$
\begin{aligned}
\phi_a &= \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \\
J_a &= \begin{pmatrix} J_+ \\ J_- \end{pmatrix} \qquad (2.31)
\end{aligned}
$$

where $a = +,-$; $\phi_+, J_+ \in C_+$ and $\phi_-, J_- \in C_-$. Furthermore, if we assume the metric in this two dimensional space to be $(1,-1)$, then we can write the generating functional for the free theory in eq. (2.24) as

$$Z[J_+, J_-] = N\, \mathrm{e}^{-\frac{i}{2}\int d^4x\, d^4y\, J^a(x) G_{ab}(x,y) J^b(y)} \tag{2.32}$$

which would represent the generating functional corresponding to an action

$$\begin{aligned} S &= S(\phi_+, J_+) - S(\phi_-, J_-) \\ &= \int d^4x \left(\mathcal{L}(\phi_+, J_+) - \mathcal{L}(\phi_-, J_-)\right) \end{aligned} \tag{2.33}$$

so that the components of the propagator can simply be written as

$$iG_{ab}(x,y) = (-i)^2 \frac{1}{Z} \frac{\delta^2 Z[J]}{\delta J^a(x) \delta J^b(y)}\bigg|_{J=0} \tag{2.34}$$

We note here that the time integration, in this case, is over the usual interval, $-\infty \leq t \leq \infty$, and that we have gotten rid of the branch, C_-, of the closed time path contour by doubling the degrees of freedom. It is immediately clear from the definition in eq. (2.30) that G_{++} corresponds to the conventional time ordered Greens function whereas G_{--} is an anti-time ordered Greens function. (Time ordering on the branch C_- corresponds to anti-time ordering in the conventional sense since time runs backwards along this branch.) Thus, intuitively, it is quite clear that the finite temperature Greens functions would display very different analyticity properties compared to the zero temperature ones. We will come to this later. But for the present, let us note that we can write the Greens functions as a 2×2 matrix as (either on the two branches of the closed time path contour or in the space of the doubled degrees of freedom)

$$G = \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix} \tag{2.35}$$

So far, we have talked about the Greens functions in the coordinate space. However, as we know from our experience at zero temperature, calculations simplify considerably in momentum space. In the real time formalism, (unlike in the imaginary time formalism) we can take the Fourier transform of the Greens functions in a simple manner and they take the form

$$\begin{aligned} G_{++}(p) &= \left(\frac{1}{p^2 - m^2 + i\epsilon} - 2i\pi n_B(|p^0|)\delta(p^2 - m^2)\right) \\ G_{+-}(p) &= -2i\pi\left(\theta(-p^0) + n_B(|p^0|)\right)\delta(p^2 - m^2) \\ G_{-+}(p) &= -2i\pi\left(\theta(p^0) + n_B(|p^0|)\right)\delta(p^2 - m^2) \\ G_{--}(p) &= \left(-\frac{1}{p^2 - m^2 - i\epsilon} - 2i\pi n_B(|p^0|)\delta(p^2 - m^2)\right) \end{aligned} \tag{2.36}$$

We note here that even though the distribution functions in the above expressions can also be written as $n_B(\omega_p)$, we write it in a form which is the most convenient for calculations. It is quite clear now that the Greens functions, even for the free theory, do not have well defined analyticity properties. This is because, by definition,

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{2i\pi}\left(\frac{1}{x - i\epsilon} - \frac{1}{x + i\epsilon}\right) \tag{2.37}$$

and the delta function is not analytic in either half of the complex plane. From the definition of the Greens functions in eqs. (2.29) and (2.36) it is clear that not all the components of this matrix are independent. In fact, it is easy to check that the matrix components of the Greens function satisfy the constraint relation

$$G_{++} + G_{--} = G_{+-} + G_{-+} \tag{2.38}$$

Thus, there are only three independent components of the matrix Greens function.

It is interesting to note here also that the real time propagators (iG_{ab}) can be given an intuitive physical picture, unlike the imaginary time ones. First, we note that the real time propagators consist of a sum of two parts – one corresponding to the propagator at zero temperature and the other corresponding to a temperature dependent part. It is to be noted that the temperature dependent part is the same for all the components of the Greens function. Second, if we look at G_{++}, we note that while the zero temperature part of the propagator can be thought of in the conventional manner as representing the exchange of a virtual particle, the temperature dependent part, as we see from eq. (2.36) represents an on-shell contribution. In a plasma of hot particles, there is a distribution of real particles and these real particles can also participate in the absorption and emission process in addition to the exchange of virtual particles. And the temperature dependent part of the Feynman propagator describes precisely this.

The other important thing to note here from the structure of the propagator is the fact that since the temperature dependent part corresponds to an on-shell contribution, it is unlikely to change the ultraviolet properties of the theory. Namely, a quantum system cannot develop any additional ultraviolet divergences at finite temperature. Thus, from the point of view of renormalization of a theory, the zero temperature counter terms will be sufficient. (This, however, does not say anything about the infrared divergences in a theory which can

become quite severe at finite temperature.) This fact can also be understood in a physical manner in the following way. The thermal contribution comes from a distribution of real particles in a hot medium. This distribution, however, is highly Boltzmann suppressed for high energies and thus temperature contributions are negligible in the ultraviolet region.

It is clear from this description that the fields as well as the propagators acquire a natural two dimensional matrix structure at finite temperature. This is a characteristic property of real time formalisms and is commonly referred to as the doubling of degrees of freedom. We will come to this in a later chapter, but let us simply note here that heuristically, one can think of this doubling as follows. At finite temperature, the entire system really consists of two parts – one the quantum system and the other the heat bath. Doubling of degrees of freedom merely represents this aspect in some sense. It is, in fact, required for the self-consistency of the formalism as we will see shortly.

Let us also note that the Greens functions that we have discussed so far, can be calculated perturbatively in the presence of interactions in a diagrammatic form. However, often, we are not interested in the causal Greens functions. Rather, in statistical systems, we are more interested in the physical Greens functions which are defined as (for bosonic fields)

$$\begin{aligned} iG_R(x,x') &= \theta(t-t')\langle\,[\phi(x),\phi(x')]\,\rangle \\ iG_A(x,x') &= \theta(t'-t)\langle\,[\phi(x),\phi(x')]\,\rangle \\ iG_C(x,x') &= \langle\,[\phi(x),\phi(x')]_+\rangle \end{aligned} \tag{2.39}$$

These are essentially the retarded, the advanced and the correlated Greens functions. It would be interesting, therefore, to see how the diagrammatic technique of finite temperature can be used to study dynamical questions which depend on the physical Greens functions. A typical example of such a study, for instance, would be the formation of domains and the growth of domain sizes during a phase transition.

The interesting point to note from the definitions in eqs. (2.30) and (2.39), however, is that the physical Greens functions can be expressed in terms of the components of the causal Greens functions as

$$\begin{aligned} G_R &= G_{++} - G_{+-} = G_{-+} - G_{--} \\ G_A &= G_{++} - G_{-+} = G_{+-} - G_{--} \\ G_C &= G_{++} + G_{--} = G_{+-} + G_{-+} \end{aligned} \tag{2.40}$$

Thus, one way to calculate the physical Greens functions using the diagrammatic techniques is as follows. We know that G_{++}, G_{+-}, G_{-+} and G_{--} can be calculated using Feynman diagrams. One can, then, obtain the physical Greens functions from these calculations using eq. (2.40). (It is worth noting here, however, that there have also been attempts at calculating the physical Greens functions using Feynman diagrams directly.) In fact, let us note that if we define a matrix of physical Greens functions as

$$\widehat{G} = \begin{pmatrix} 0 & G_A \\ G_R & G_C \end{pmatrix} \tag{2.41}$$

then, from eq. (2.40) we see that we can write

$$
\begin{aligned}
G &= Q^{-1}\,\widehat{G}\,Q \\
\widehat{G} &= Q\,G\,Q^{-1}
\end{aligned}
\tag{2.42}
$$

where the unitary matrix relating the two is given by

$$
\begin{aligned}
Q &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
Q^{-1} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = Q^{\dagger}
\end{aligned}
\tag{2.43}
$$

Thus, we note that once the matrix G has been calculated diagrammatically, the physical Greens functions, $\widehat{G}$, can be obtained in a simple manner through a change of basis defined by eq. (2.42).

For the sake of completeness, we note here the forms of the retarded, advanced and correlated Greens functions at finite temperature for the free Klein-Gordon theory which can be obtained from eqs. (2.38) and (2.40).

$$
\begin{aligned}
G_R(p) &= \frac{1}{p^2 - m^2 + i\epsilon p^0} = \frac{1}{(p^0 + i\epsilon/2)^2 - \omega_p^2} \\
G_A(p) &= \frac{1}{p^2 - m^2 - i\epsilon p^0} = \frac{1}{(p^0 - i\epsilon/2)^2 - \omega_p^2} \\
G_C(p) &= -2i\pi\left(1 + 2n_B(|p^0|)\right)\delta(p^2 - m^2)
\end{aligned}
\tag{2.44}
$$

It should be noted that the retarded and the advanced Greens functions are independent of the temperature at the lowest order as they should be. (The retarded and the advanced Greens functions are described in terms of ensemble averages of commutators (see eq. (2.39))

which are c–number functions.) The effect of the boundary conditions or the initial state conditions are carried completely by the correlated Greens function. Furthermore, the retarded Greens function is analytic in the upper half of the complex p^0–plane whereas the advanced Greens function is analytic in the lower half plane much like at zero temperature. But the correlated Greens function does not have any well defined analytic behavior.

Let us record here, without going into details, that the form of the finite temperature Greens functions for a free Dirac field can also be worked out in a straightforward manner and have the forms

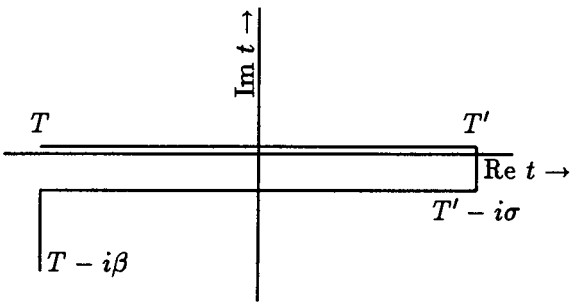
$$\begin{aligned}
S_{++}(p) &= (\gamma^\mu p_\mu + m)\left(\frac{1}{p^2 - m^2 + i\epsilon} + 2i\pi n_F(|p^0|)\delta(p^2 - m^2)\right) \\
S_{+-}(p) &= 2i\pi(\gamma^\mu p_\mu + m)\left(n_F(|p^0|) - \theta(-p^0)\right)\delta(p^2 - m^2) \\
S_{-+}(p) &= 2i\pi(\gamma^\mu p_\mu + m)\left(n_F(|p^0|) - \theta(p^0)\right)\delta(p^2 - m^2) \qquad (2.45) \\
S_{--}(p) &= (\gamma^\mu p_\mu + m)\left(-\frac{1}{p^2 - m^2 - i\epsilon} + 2i\pi n_F(|p^0|)\delta(p^2 - m^2)\right)
\end{aligned}$$

where the fermion distribution function is given by

$$n_F(|p^0|) = \frac{1}{e^{\beta|p^0|} + 1}$$

The physical Greens functions can, then, be obtained from these in a straightforward manner using the unitary matrix, Q, in eq. (2.43).

To end this section, let us note that one can generalize the closed time path that we have discussed so far in the following manner.



Here, we assume that $0 \leq \sigma \leq \beta$. For such a contour, once again, the Greens functions can be easily obtained from eq. (2.29) and, for the scalar field, have the form in the momentum space (see and compare with eq. (2.36))

$$\begin{aligned} G_{11}(p) &= G_{++}(p) \\ G_{12}(p) &= e^{\sigma p^0} G_{+-}(p) \\ G_{21}(p) &= e^{-\sigma p^0} G_{-+}(p) \\ G_{22}(p) &= G_{--}(p) \end{aligned} \tag{2.46}$$

where we have labelled the paths (and, therefore, the indices of the Greens functions) differently to bring out the difference. A very common choice for σ turns out to be $\sigma = \frac{\beta}{2}$. In particular, we note that, in this case,

$$\begin{aligned} G_{12}(p) &= -2i\pi\, e^{\beta p^0/2} \left(\theta(-p^0) + n_B(|p^0|)\right) \delta(p^2 - m^2) \\ &= -2i\pi\, e^{\beta p^0/2} \left((1 + n_B(|p^0|))\theta(-p^0) + n_B(|p^0|)\theta(p^0)\right) \\ &\quad \times \delta(p^2 - m^2) \end{aligned}$$

$$= -2i\pi\, e^{\beta|p^0|/2}\, n_B(|p^0|)\, \delta(p^2 - m^2) \tag{2.47}$$

$$\begin{aligned} G_{21}(p) &= -2i\pi\, e^{-\beta p^0/2}\, (\theta(p^0) + n_B(|p^0|))\, \delta(p^2 - m^2) \\ &= -2i\pi\, e^{-\beta p^0/2}\, ((1 + n_B(|p^0|))\theta(p^0) + n_B(|p^0|)\theta(-p^0)) \\ &\qquad \times \delta(p^2 - m^2) \\ &= -2i\pi\, e^{\beta|p^0|/2}\, n_B(|p^0|)\, \delta(p^2 - m^2) \end{aligned}$$

Namely, with this choice for σ,

$$G_{12}(p) = G_{21}(p)$$

As we will see in the next chapter, this choice of the path and the propagators also coincides with that of the thermofield dynamics formalism.

2.4 Analyticity Properties

As we have discussed earlier, the Greens functions do not have very well defined analyticity properties (remember the delta function, for example). Let us investigate this in some more detail with the simple example of the Greens functions of the free scalar theory although similar conclusions can be easily derived for the ensemble averages of any two arbitrary operators which we will do in the next chapter. We note from eq. (2.38) that

$$\begin{aligned} \mathrm{Re}\, G_{++}(p) &= P\, \frac{1}{p^2 - m^2} \\ \mathrm{Im}\, G_{++}(p) &= -\pi\, \coth\left(\frac{\beta|p^0|}{2}\right) \delta(p^2 - m^2) \end{aligned} \tag{2.48}$$

where P stands for the principal part and eq. (2.48) leads to the dispersion relation

$$\mathrm{Re}\, G_{++}(p^0,\vec{p}) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dp'^0 \, \frac{\mathrm{Im}\, G_{++}(p'^0,\vec{p})}{p^0 - p'^0} \tanh\left(\frac{\beta p'^0}{2}\right) \quad (2.49)$$

Once again this shows that G_{++} does not have well defined analyticity properties.

Similarly, we note from eq. (2.44) that

$$\begin{aligned} \mathrm{Re}\, G_R(p) &= P \frac{1}{p^2 - m^2} \\ \mathrm{Im}\, G_R(p) &= -\pi\, \mathrm{sgn}(p^0)\, \delta(p^2 - m^2) \end{aligned} \quad (2.50)$$

It immediately follows from eqs. (2.48) and (2.50) that

$$\begin{aligned} \mathrm{Re}\, G_R(p) &= \mathrm{Re}\, G_{++}(p) \\ \mathrm{Im}\, G_R(p) &= \tanh\left(\frac{\beta p^0}{2}\right) \mathrm{Im}\, G_{++}(p) \end{aligned} \quad (2.51)$$

Once again, since the retarded Greens function is analytic in the upper half of the complex p^0–plane, this shows that G_{++} is not analytic in either half of the planes (This is also evident from the fact that $\delta(x)$ is not analytic in either half of the complex x-plane.). We note here that even though we have derived these relations for the free Greens functions (propagators), they are true also in an interacting theory as the derivation in the next chapter will show.

We can derive similar relations for the advanced Greens function as well. However, the more interesting question to ask is the

relation between the Greens functions in the imaginary and the real time formalisms. This can be obtained in the following way. From the form of the Greens function in eq. (1.41), we note that

$$\mathcal{G}_\beta(p^0,\vec{p}) = \frac{1}{(p^0)^2 + \vec{p}^2 + m^2}$$

where p^0 is assumed to take discrete values as discussed earlier. We note now that if we treat p^0 as a continuous variable and analytically continue it to the complex p^0–plane, then, we obtain

$$\begin{aligned} \mathcal{G}_\beta(i(p^0 + i\epsilon),\vec{p}) &= -\frac{1}{(p^0 + i\epsilon)^2 - \vec{p}^2 - m^2} \\ &= -\frac{1}{p^2 - m^2 + 2i\epsilon p^0} \\ &= -G_R(p) \end{aligned} \tag{2.52}$$

This shows that if we know the values of the imaginary time Greens functions at a set of discrete points, then by analytic continuation, we can obtain the real time, retarded Greens function which then would lead to the real time Feynman Greens function through the relation in eq. (2.51). The analytic continuation of a function, whose values are known only at a set of discrete values on the imaginary axis in the complex energy plane, to the real axis, however, is not unambiguous in general. For example, we can always multiply a function by $e^{\beta\omega}$ which has unit value at $\omega = 2in\pi/\beta$. However, if the functions are well behaved and convergent at infinity, then according to a theorem in complex analysis, the analytic continuation is unique. This, therefore, establishes the connection between the Greens functions in the two formalisms.

2.5 One Loop Propagator

Let us now work out the propagator for a self-interacting ϕ^4–theory up to one loop at finite temperature. As we have discussed earlier, the complete action that enters the generating functional in eqs. (2.18) and (2.20) can be written as (see eq. (2.33))

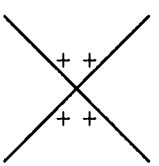
$$S = \int d^4x \, (\mathcal{L}(\phi_+) - \mathcal{L}(\phi_-)) \tag{2.53}$$

where

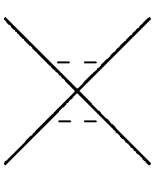
$$\mathcal{L}(\phi) = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\lambda}{4!}\,\phi^4 \tag{2.54}$$

As before we can think of ϕ_+ and ϕ_- to correspond to the values of the fields on the two branches of the contour or simply as two components of a thermal doublet of scalar fields.

We have already derived the propagator for the free theory in eq. (2.36). The interaction vertices can also be read out from the action in eq. (2.53).

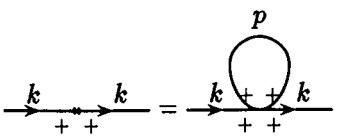


$= -i\lambda$



$= i\lambda$

It is clear that there are two kinds of vertices in the theory now – one involving the ϕ_+ fields while the other involves ϕ_- fields. The strength of interaction is the same for the two – only the signs are opposite. Given this, we can immediately calculate the one loop corrections to the theory which would only involve a mass correction.



In fact, note that

$$\begin{aligned} -i\Delta m_+^2 &= \frac{(-i\lambda)}{2}\int \frac{d^4p}{(2\pi)^4}\, iG_{++}(p) \\ &= \frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\left(\frac{1}{p^2-m^2+i\epsilon} - 2i\pi n_B(|p^0|)\delta(p^2-m^2)\right) \\ &= -i\left(\Delta m_0^2 + \Delta m_T^2\right) \end{aligned} \tag{2.55}$$

The zero temperature and the finite temperature corrections to the mass correction are clear now. It is also clear that the only divergent integral is the temperature independent one and can be taken care of with a zero temperature counter term.

Let us note next that if we do the p^0 integral using the method of residues, then the zero temperature part takes the form

$$-i\Delta m_0^2 = -\frac{i\lambda}{4}\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\omega_p} \tag{2.56}$$

whereas the p^0 integral in the temperature dependent part can be trivially done using the delta function and gives

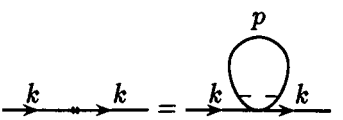
$$-i\Delta m_T^2 = -\frac{i\lambda}{2}\int \frac{d^3p}{(2\pi)^3}\,\frac{n_B(\omega_p)}{\omega_p} \tag{2.57}$$

The total mass correction at finite temperature, therefore, becomes, using eq. (1.60),

$$\Delta m_+^2 = \Delta m_0^2 + \Delta m_T^2 = \frac{\lambda}{4}\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\omega_p}\coth\left(\frac{\beta\omega_p}{2}\right) \tag{2.58}$$

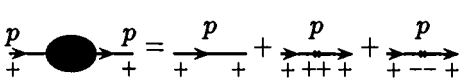
This is precisely the value obtained for the mass correction using the imaginary time formalism (see eq. (1.47)).

For the ϕ_- field, similarly, we can calculate and show that



$$\Delta m_-^2 = \frac{i\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\,G_{--}(p) = \Delta m_+^2 \tag{2.59}$$

Given this, we can now calculate the one loop correction to the propagator, which would involve both + and − type vertices,



and it is clear that it will have the form

$$
\begin{aligned}
iG^{(1)}_{++}(p) &= iG_{++} + (-i\Delta m^2_+)(iG_{++})^2 + (i\Delta m^2_-)(iG_{+-})(iG_{-+}) \\
\text{or, } G^{(1)}_{++}(p) &= G_{++}(p) + \Delta m^2_+ \left((G_{++}(p))^2 - G_{+-}(p)G_{-+}(p)\right)
\end{aligned}
\tag{2.60}
$$

Let us note from the structure of the Greens functions in eq. (2.38) that each propagator involves a delta function, the square of which is an ill defined quantity. Consequently, eq. (2.60) would appear to be ill defined. However, such ill defined quantities cancel out in the expression (2.60) giving rise to a meaningful perturbative expansion. This is another way to see that a doubling of degrees of freedom is really necessary for the consistency of a meaningful diagrammatic description. It is worth noting here that some real time formalisms which did not involve a doubling failed for this particular reason. Using a regularized expression for the delta function as in eq. (2.37) (We will discuss more on this later.), eq. (2.60) can also be written as

$$
G^{(1)}_{++}(p) = G_{++}(p) + \Delta m^2_+ \frac{\partial G_{++}(p)}{\partial m^2} \tag{2.61}
$$

which shows that the mass correction, merely shifts the position of the pole in the propagator much like at zero temperature.

2.6 References

Aurenche, P and **T. Becherray**, Nuc. Phys. **B379** (1992) 259.

Bakshi, P.M and **K.T. Mahanthappa**, J. Math. Phys. **4** (1963) 1.

Baym, G and **N. Mermin**, J. Math. Phys. **2** (1961) 232.

Bedaque, P.F and **A. Das**, Mod. Phys. Lett. **A8** (1993) 3151.

Chou, K.C *et al*, Phys. Reports **118** (1985)1.

van Eijck, M.A and **Ch. G. van Weert**, Phys. Lett. **B278** (1992) 305.

van Eijck, M.A, R. Kobes and **Ch. G. van Weert**, Phys. Rev. **D50** (1994) 4097.

Keldysh, L.V, Soviet Physics *JETP* **20** (1965) 1018.

Schwinger, J, J. Math. Physics **2** (1961) 407.

Schwinger, J, *Lecture Notes of Brandeis Summer Institute in Theoretical Physics* (1960).

Chapter 3

Thermofield Dynamics

Another very interesting approach to thermal field theories within the framework of real time formalism is thermofield dynamics. Thermofield dynamics has already been widely used in many areas of physics. We can essentially think of thermofield dynamics as an operator formalism (although a path integral representation does exist for it and will be described later in this chapter) and complementary to the closed time path formalism. It can answer such questions as the structure of the thermal vacuum, the nature of the Goldstone states and so on that are not possible within the framework of the closed time path formalism. It can also describe quite naturally the time development of quantities near equilibrium.

3.1 General Formalism

To introduce thermofield dynamics, let us recall (see eq. (1.3)) that the ensemble average of any operator in thermal equilibrium is given by ($Z(\beta)$ is the partition function of the system.)

$$\langle A\rangle_\beta = Z^{-1}(\beta)\,\mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}}A \tag{3.1}$$

and can be written also as

$$\langle A\rangle_\beta = Z^{-1}(\beta)\sum_n \mathrm{e}^{-\beta E_n}\langle n|A|n\rangle \tag{3.2}$$

where we are assuming that the eigenvalues of $\mathcal{H}$ are discrete for simplicity and that

$$\begin{aligned}\mathcal{H}|n\rangle &= E_n|n\rangle \\ \langle n|m\rangle &= \delta_{nm}\end{aligned} \tag{3.3}$$

The partition function, which has the form,

$$Z(\beta) = \mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}}$$

resembles the vacuum to vacuum transition amplitude (or the generating functional) of zero temperature field theory, but does not coincide with it. We can take over the machinery of the zero temperature field theory completely if we can express the ensemble average of any operator as an expectation value in a vacuum (say, a thermal vacuum). Thus, if we can define a state $|0,\beta\rangle$ such that we can write

$$\langle A\rangle_\beta = \langle 0,\beta|A|0,\beta\rangle = Z^{-1}(\beta)\sum_n \mathrm{e}^{-\beta E_n}\langle n|A|n\rangle \tag{3.4}$$

for any arbitrary operator A, then, the finite temperature formalism will be completely parallel to the zero temperature one.

Let us next assume that we can express the state $|0,\beta\rangle$ as a linear superposition of the states of our Hilbert space, namely, let

$$|0,\beta\rangle = \sum_n |n\rangle\langle n|0,\beta\rangle = \sum_n f_n(\beta)\,|n\rangle \tag{3.5}$$

where we have used the completeness of the states of our Hilbert space. Then, we obtain

$$\langle 0,\beta|A|0,\beta\rangle = \sum_{n,m} f_n^*(\beta) f_m(\beta)\langle n|A|m\rangle \tag{3.6}$$

This will agree with eq. (3.4) provided

$$f_n^*(\beta) f_m(\beta) = Z^{-1}(\beta)\,\mathrm{e}^{-\beta E_n}\delta_{nm} \tag{3.7}$$

However, as we see from eq. (3.5), $f_n(\beta)$'s are ordinary numbers, and, therefore, it is not possible to satisfy eq. (3.7). This shows that as long as we restrict ourselves to the original Hilbert space, we cannot define a finite temperature vacuum which would satisfy eq. (3.4).

Let us note, however, that the condition, eq. (3.7), is quite analogous to an orthonormality condition for state vectors. But, $f_n(\beta)$'s are not state vectors. Therefore, to carry this program through, let us introduce a fictitious system which is an identical copy of the original system we are interested in and denote it by a tilde system. Let

us consider the product space of the two systems as (The state $|\tilde{n}\rangle$ is a tilde state with the number eigenvalue n.)

$$|n, \tilde{m}\rangle = |n\rangle \otimes |\tilde{m}\rangle$$

and expand the thermal vacuum as

$$|0, \beta\rangle = \sum_n f_n(\beta)|n, \tilde{n}\rangle = \sum_n f_n(\beta)|n\rangle \otimes |\tilde{n}\rangle \tag{3.8}$$

In this case, we note that

$$\begin{aligned} \langle 0, \beta|A|0, \beta\rangle &= \sum_{n,m} f_n^*(\beta) f_m(\beta) \langle n, \tilde{n}|A|m, \tilde{m}\rangle \\ &= \sum_{n,m} f_n^*(\beta) f_n(\beta) \langle n|A|n\rangle \, \delta_{nm} \\ &= \sum_n f_n^*(\beta) f_n(\beta) \langle n|A|n\rangle \end{aligned} \tag{3.9}$$

Here, we have used the fact that an operator of the original physical system does not act on the Hilbert space of the tilde system and vice versa so that using the orthonormality of the states, we can write

$$\langle n, \tilde{m}|A|n', \tilde{m}'\rangle = \langle n|A|n'\rangle \, \langle \tilde{m}|\tilde{m}'\rangle = \delta_{mm'} \langle n|A|n'\rangle \tag{3.10}$$

Similarly, we can write for a tilde operator

$$\langle n, \tilde{m}|\tilde{A}|n', \tilde{m}'\rangle = \langle n|n'\rangle \, \langle \tilde{m}|\tilde{A}|\tilde{m}'\rangle = \delta_{nn'} \langle \tilde{m}|\tilde{A}|\tilde{m}'\rangle \tag{3.11}$$

We note now from eq. (3.9) that this can represent an ensemble average provided (see eq. (3.4))

$$f_n^*(\beta) f_n(\beta) = Z^{-1}(\beta) \mathrm{e}^{-\beta E_n} \tag{3.12}$$

which can have the simple real solution

$$f_n(\beta) = f_n^*(\beta) = Z^{-1/2}(\beta)\, \mathrm{e}^{-\frac{\beta E_n}{2}} \tag{3.13}$$

This, therefore, shows that we can introduce a temperature dependent vacuum state so that the statistical ensemble average of any operator can be identified with the expectation value of the operator in this state. This, however, entails a doubling of the Hilbert space. (This doubling is quite reminiscent of our discussion of doubling, in the last chapter, within the context of the closed time path formalism.) The advantage, on the other hand, lies in the fact that the techniques of zero temperature field theory can now be carried over to the finite temperature case. We would like to note here that choices other than the simple one in eq. (3.13) are, indeed, possible. However, we would continue with our simple choice in what follows.

3.2 Fermionic Oscillator

Let us now go into the details of the formalism through simple examples. To start with, let us consider a fermionic oscillator with frequency ω described by the Hamiltonian ($\hbar = 1$)

$$H = \omega\, a^\dagger a \tag{3.14}$$

The creation and the annihilation operators of the theory satisfy the anti-commutation relations

$$\begin{aligned} [a, a^\dagger]_+ &= 1 \\ [a, a]_+ &= [a^\dagger, a^\dagger]_+ = 0 \end{aligned} \tag{3.15}$$

The Hilbert space is two dimensional with the states given by $|0\rangle$ and $|1\rangle = a^\dagger|0\rangle$.

According to our general discussion of thermofield dynamics, we are next supposed to introduce a tilde system which is identical to the original system. Thus, we define

$$\tilde{H} = \omega \tilde{a}^\dagger \tilde{a} \tag{3.16}$$

where the anti-commutation relations satisfied by the tilde creation and annihilation operators are given by

$$\begin{aligned} [\tilde{a}, \tilde{a}^\dagger]_+ &= 1 \\ [\tilde{a}, \tilde{a}]_+ &= [\tilde{a}^\dagger, \tilde{a}^\dagger]_+ = 0 \end{aligned} \tag{3.17}$$

Furthermore, we assume that the creation and annihilation operators for the tilde and the non-tilde systems anti-commute.

The product space of states for the two systems now becomes four dimensional with the basis states

$$|0, \tilde{0}\rangle;\ |0, \tilde{1}\rangle;\ |1, \tilde{0}\rangle;\ |1, \tilde{1}\rangle \tag{3.18}$$

We assume the energy eigenstates to be orthonormal and according to our discussions, we can choose for this simple example (see eq. (3.8))

$$|0,\beta\rangle = f_0(\beta)|0\rangle \otimes |\tilde{0}\rangle + f_1(\beta)|1\rangle \otimes |\tilde{1}\rangle \tag{3.19}$$

The normalization of the state $|0,\beta\rangle$ gives a condition on the coefficients of expansion

$$\langle 0,\beta|0,\beta\rangle = |f_0(\beta)|^2 + |f_1(\beta)|^2 = 1 \tag{3.20}$$

We derive a second condition by requiring that the state $|0,\beta\rangle$ gives the right thermal distribution function

$$\langle 0,\beta|N|0,\beta\rangle = \langle 0,\beta|a^\dagger a|0,\beta\rangle = |f_1(\beta)|^2 = \frac{1}{e^{\beta\omega}+1} \tag{3.21}$$

From eqs. (3.20) and (3.21), we can then choose

$$\begin{aligned} f_0(\beta) = f_0^*(\beta) &= \frac{1}{\sqrt{1+e^{-\beta\omega}}} \\ f_1(\beta) = f_1^*(\beta) &= \frac{e^{-\beta\omega/2}}{\sqrt{1+e^{-\beta\omega}}} \end{aligned} \tag{3.22}$$

which should be compared with eq. (3.12).

Thus, we see, from eq. (3.19), that the thermal vacuum, in this case, can be written as

$$|0,\beta\rangle = \frac{1}{\sqrt{1+e^{-\beta\omega}}}\left(|0,\tilde{0}\rangle + e^{-\beta\omega/2}|1,\tilde{1}\rangle\right) \tag{3.23}$$

We also observe that, in this case, it follows trivially from the anti-commutation relations of the operators that

$$\begin{aligned} \left(\tilde{a}a - a^\dagger\tilde{a}^\dagger\right)^{2n}|0,\tilde{0}\rangle &= (-1)^n|0,\tilde{0}\rangle \\ \left(\tilde{a}a - a^\dagger\tilde{a}^\dagger\right)^{2n+1}|0,\tilde{0}\rangle &= (-1)^{n+1}|1,\tilde{1}\rangle \end{aligned} \tag{3.24}$$

Therefore, we note that we can define a Hermitian operator

$$Q(\theta) = -i\theta(\beta)\left(\tilde{a}a - a^\dagger\tilde{a}^\dagger\right) \tag{3.25}$$

where $\theta(\beta)$ is a real parameter such that the operator

$$U(\theta) = e^{-iQ(\theta)} = \mathrm{e}^{-\theta(\beta)\left(\tilde{a}a - a^\dagger\tilde{a}^\dagger\right)} \tag{3.26}$$

would be formally unitary. Furthermore, this operator would generate the thermal vacuum from the vacuum of the doubled space as

$$U(\theta)|0,\tilde{0}\rangle = \cos\theta(\beta)\,|0,\tilde{0}\rangle + \sin\theta(\beta)\,|1,\tilde{1}\rangle = |0,\beta\rangle \tag{3.27}$$

provided we identify

$$\cos\theta(\beta) = \frac{1}{\sqrt{1+\mathrm{e}^{-\beta\omega}}} \qquad \sin\theta(\beta) = \frac{\mathrm{e}^{-\beta\omega/2}}{\sqrt{1+\mathrm{e}^{-\beta\omega}}} \tag{3.28}$$

Thus, we conclude that the thermal vacuum and the vacuum of the doubled space are related by the formally unitary transformation of eq. (3.26).

It is also straightforward to calculate the change in the operators under this unitary transformation. From the definition

$$A(\beta) = U(\theta)\,A\,U^\dagger(\beta)$$

for any operator A, we obtain the transformation of the creation and the annihilation operators to be

$$
\begin{aligned}
a(\beta) &= a\cos\theta(\beta) - \tilde{a}^\dagger \sin\theta(\beta) \\
\tilde{a}(\beta) &= \tilde{a}\cos\theta(\beta) + a^\dagger \sin\theta(\beta) \\
a^\dagger(\beta) &= a^\dagger \cos\theta(\beta) - \tilde{a}\sin\theta(\beta) \\
\tilde{a}^\dagger(\beta) &= \tilde{a}^\dagger \cos\theta(\beta) + a\sin\theta(\beta)
\end{aligned} \tag{3.29}
$$

where $\theta(\beta)$ is given in eq. (3.28). Let us note here, for later use, that if we define a doublet as

$$
A = \begin{pmatrix} a \\ \tilde{a}^\dagger \end{pmatrix} \tag{3.30}
$$

then, we can write the thermal transformations of eq. (3.29) also as

$$
\begin{aligned}
A(\beta) &= \begin{pmatrix} a(\beta) \\ \tilde{a}^\dagger(\beta) \end{pmatrix} = U(\theta) A U^\dagger(\theta) = \overline{U}(\theta)\, A \\
&= \begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix} \begin{pmatrix} a \\ \tilde{a}^\dagger \end{pmatrix}
\end{aligned} \tag{3.31}
$$

with

$$
\overline{U}(\theta) = \begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix} \tag{3.32}
$$

It follows now from eqs. (3.27)-(3.29) that

$$
\begin{aligned}
a(\beta)|0,\beta\rangle &= (a\cos\theta(\beta) - \tilde{a}^\dagger \sin\theta(\beta))\,|0,\beta\rangle \\
&= (a\cos\theta(\beta) - \tilde{a}^\dagger \sin\theta(\beta))(\cos\theta(\beta)|0,\tilde{0}\rangle + \sin\theta(\beta)|1,\tilde{1}\rangle) \\
&= \cos\theta(\beta)\sin\theta(\beta)\,(a|1,\tilde{1}\rangle - \tilde{a}^\dagger|0,\tilde{0}\rangle) \\
&= 0
\end{aligned}
$$

$$\begin{aligned}
\tilde{a}(\beta)|0,\beta\rangle &= (\tilde{a}\cos\theta(\beta) + a^\dagger \sin\theta(\beta))|0,\beta\rangle \\
&= (\tilde{a}\cos\theta(\beta) + a^\dagger \sin\theta(\beta))(\cos\theta(\beta)|0,\tilde{0}\rangle + \sin\theta(\beta)|1,\tilde{1}\rangle) \\
&= \cos\theta(\beta)\sin\theta(\beta)\,(\tilde{a}|1,\tilde{1}\rangle + a^\dagger|0,\tilde{0}\rangle) \\
&= 0 \qquad (3.33)
\end{aligned}$$

where we have used the fact that the tilde and the nontilde operators anti-commute. This shows that the thermal vacuum is truly the vacuum state of the thermal creation and annihilation operators.

The thermal creation and annihilation operators can be easily shown to satisfy the same anti-commutation relations as the original ones since they are unitarily related. We can now construct the thermal Hilbert space and the four dimensional space would consist of the states

$$|0,\beta\rangle;\ a^\dagger(\beta)|0,\beta\rangle;\ \tilde{a}^\dagger(\beta)|0,\beta\rangle;\ a^\dagger(\beta)\tilde{a}^\dagger(\beta)|0,\beta\rangle$$

It is worth noting from the definition of the thermal vacuum in eq. (3.27) that it is like a squeezed state. However, there is an essential difference. A little bit of analysis will show that the thermal Hilbert space and the original Hilbert space will not become isomorphic when the space contains an infinite number of modes. In other words, the transformation in eq. (3.26), even though formally unitary, is more like a Bogoliubov transformation. It is also worth noting from eq. (3.33) that annihilating a particle quantum in the thermal vacuum is equivalent to creating a tilde quantum and vice versa. This allows us to intuitively think of the tilde particles as kind of hole states of the particle or particle

states of the heat bath which gives an intuitive meaning to the doubling of the degrees of freedom in thermofield dynamics.

The states in the thermal Hilbert space are not eigenstates of either H or $\tilde{H}$ as can be easily checked. However, from eq. (3.29), it is clear that

$$a^\dagger(\beta)a(\beta) - \tilde{a}^\dagger(\beta)\tilde{a}(\beta) = a^\dagger a - \tilde{a}^\dagger\tilde{a} \tag{3.34}$$

Thus, if we define a total Hamiltonian for the combined system as

$$\hat{H} = H - \tilde{H} \tag{3.35}$$

then, it is clear that the thermal states will be eigenstates of $\hat{H}$.

3.3 Bosonic Oscillator

Let us next summarize, without going into much detail, the results of thermofield dynamics when applied to a bosonic oscillator. The Hamiltonian, in this case, has the same form as in eq. (3.14)

$$H = \omega\, a^\dagger a$$

except that the creation and the annihilation operators satisfy commutation relations

$$\begin{aligned} [a, a^\dagger] &= 1 \\ [a, a] &= [a^\dagger, a^\dagger] = 0 \end{aligned} \tag{3.36}$$

The Hilbert space is infinite dimensional and the eigenstates of the Hamiltonian satisfy

$$H|n\rangle = n\,\omega\,|n\rangle \qquad n = 0, 1, 2, \cdots \tag{3.37}$$

Following the discussion of thermofield dynamics, we now introduce the tilde system described by the Hamiltonian

$$\tilde{H} = \omega \tilde{a}^{\dagger} \tilde{a}$$

where the tilde creation and annihilation operators satisfy identical commutation relations as in eq. (3.36) and are assumed to commute with the original operators. We can now represent a typical state in the product space as

$$|n, \tilde{m}\rangle = |n\rangle \otimes |\tilde{m}\rangle \qquad n, m = 0, 1, 2, \cdots$$

where the states are assumed to be normalized. The thermal vacuum can now be shown, as in the last section, to be expressible as

$$|0, \beta\rangle = \sqrt{1 - \mathrm{e}^{-\beta\omega}} \sum_{n=0}^{\infty} \mathrm{e}^{-n\beta\omega/2} \, |n, \tilde{n}\rangle \tag{3.38}$$

which can be compared with eqs. (3.8) and (3.13).

Once again, as in the fermionic case, we can define a Hermitian operator

$$Q(\theta) = -i\theta(\beta) \left(\tilde{a} a - a^{\dagger} \tilde{a}^{\dagger} \right) \tag{3.39}$$

and show that the formally unitary operator

$$U(\theta) = \mathrm{e}^{-iQ(\theta)} = \mathrm{e}^{-\theta(\beta)\left(\tilde{a}a - a^\dagger \tilde{a}^\dagger\right)} \tag{3.40}$$

takes us to the thermal vacuum through a Bogoliubov transformation of the form

$$|0,\beta\rangle = U(\theta)|0,\tilde{0}\rangle = \mathrm{e}^{-\theta(\beta)\left(\tilde{a}a - a^\dagger \tilde{a}^\dagger\right)}|0,\tilde{0}\rangle \tag{3.41}$$

provided we identify

$$\cosh\theta(\beta) = \frac{1}{\sqrt{1-\mathrm{e}^{-\beta\omega}}} \qquad \sinh\theta(\beta) = \frac{\mathrm{e}^{-\beta\omega/2}}{\sqrt{1-\mathrm{e}^{-\beta\omega}}} \tag{3.42}$$

The thermal operators can also be obtained in a standard manner. We simply note here (see eqs. (3.30)-(3.31)) that if we define a doublet as

$$A = \begin{pmatrix} a \\ \tilde{a}^\dagger \end{pmatrix} \tag{3.43}$$

then, we can write

$$\begin{aligned} A(\beta) &= \begin{pmatrix} a(\beta) \\ \tilde{a}^\dagger(\beta) \end{pmatrix} = U(\theta) A U^\dagger(\theta) = \overline{U}(\theta)\, A \\ &= \begin{pmatrix} \cosh\theta(\beta) & -\sinh\theta(\beta) \\ -\sinh\theta(\beta) & \cosh\theta(\beta) \end{pmatrix} \begin{pmatrix} a \\ \tilde{a}^\dagger \end{pmatrix} \end{aligned} \tag{3.44}$$

with $\theta(\beta)$ given in eq. (3.42) and

$$\overline{U}(\theta) = \begin{pmatrix} \cosh\theta(\beta) & -\sinh\theta(\beta) \\ -\sinh\theta(\beta) & \cosh\theta(\beta) \end{pmatrix} \begin{pmatrix} a \\ \tilde{a}^\dagger \end{pmatrix} \tag{3.45}$$

From the definition of the thermal creation and annihilation operators, then, it can be readily verified that the thermal vacuum satisfies

$$\begin{aligned} a(\beta)|0,\beta\rangle &= (a\cosh\theta(\beta) - \tilde{a}^\dagger \sinh\theta(\beta))|0,\beta\rangle = 0 \\ \tilde{a}(\beta)|0,\beta\rangle &= (\tilde{a}\cosh\theta(\beta) - a^\dagger \sinh\theta(\beta))|0,\beta\rangle = 0 \end{aligned} \tag{3.46}$$

We see again that the state $|0,\beta\rangle$ is annihilated by the thermal annihilation operators and, in that sense, can be thought of as the thermal vacuum. Furthermore, as in the fermionic case, we note that the annihilation of a particle quantum in the thermal vacuum is equivalent to the creation of a tilde quantum. Finally, let us note that we can build up the thermal Hilbert space by applying thermal creation operators on the state $|0,\beta\rangle$. However, the states in this space will not be eigenstates of either H or $\tilde{H}$. Rather, they can be identified with the eigenstates of the operator $\hat{H} = H - \tilde{H}$ which can be thought of as the total Hamiltonian governing the dynamics of the combined system.

Let us now observe the following. For either the bosonic or the fermionic case, we note that

$$\hat{H}|0,\beta\rangle = 0 \tag{3.47}$$

Moreover, in either case, we have

$$\begin{aligned} [\hat{H}, a] &= -\omega a \\ [\hat{H}, \tilde{a}] &= \omega \tilde{a} \\ [\hat{H}, a^\dagger] &= \omega a^\dagger \\ [\hat{H}, \tilde{a}^\dagger] &= -\omega \tilde{a}^\dagger \end{aligned} \tag{3.48}$$

We also note from eqs. (3.28) and (3.42) that for the fermionic and the bosonic case, we have, respectively

$$\begin{aligned} \tan\theta(\beta) &= e^{-\beta\omega/2} \\ \tanh\theta(\beta) &= e^{-\beta\omega/2} \end{aligned} \tag{3.49}$$

From these relations, we note that we can write the conditions in eqs. (3.33) and (3.46) in a unified way as

$$\begin{aligned} a|0,\beta\rangle &= e^{-\beta\omega/2}\,\tilde{a}^\dagger|0,\beta\rangle = e^{\beta\widehat{H}/2}\,\tilde{a}^\dagger|0,\beta\rangle \\ \tilde{a}|0,\beta\rangle &= (-1)^{|a|}\,e^{-\beta\omega/2}\,a^\dagger|0,\beta\rangle = (-1)^{|a|}\,e^{-\beta\widehat{H}/2}\,a^\dagger|0,\beta\rangle \end{aligned} \tag{3.50}$$

Here $|a|$ represents the Grassmann parity of the operator a which is also the same as that of $\tilde{a}$ and takes values 0 or 1 for bosonic and fermionic operators respectively. There are also the adjoint conditions which follow in a straightforward manner. Together, they lead to what are known as the tilde conjugation relations that are essential for thermofield dynamics as we will see later in this chapter.

3.4 Free Schrödinger Field Theory

In the last two sections, we studied simple quantum mechanical systems at finite temperature in the formalism of thermofield dynamics. We will now look at some of the field theories at finite temperature in this formalism. Let us consider the free Schrödinger field theory in $3+1$ dimensions as a first example. As we know, the Lagrangian density is given by

$$\mathcal{L} = i\psi^\dagger\dot{\psi} - \frac{1}{2m}\vec{\nabla}\psi^\dagger \cdot \vec{\nabla}\psi \tag{3.51}$$

According to thermofield dynamics, we should introduce a tilde system and let us take the Lagrangian density for the tilde system to be

$$\tilde{\mathcal{L}} = -i\tilde{\psi}^\dagger\dot{\tilde{\psi}} - \frac{1}{2m}\vec{\nabla}\tilde{\psi}^\dagger \cdot \vec{\nabla}\tilde{\psi} \tag{3.52}$$

The reason for the negative sign in the first term in $\tilde{\mathcal{L}}$ will be clear shortly. From the discussion of the last chapter, we recognize that we can take the total Lagrangian density for the system to be

$$\hat{\mathcal{L}} = \mathcal{L} - \tilde{\mathcal{L}} \tag{3.53}$$

The momentum conjugate to the field variables ψ and $\tilde{\psi}$ are given by

$$\begin{aligned} \Pi(\vec{x},t) &= \frac{\partial\hat{\mathcal{L}}}{\partial\dot{\psi}(\vec{x},t)} = i\psi^\dagger(\vec{x},t) \\ \tilde{\Pi}(\vec{x},t) &= \frac{\partial\hat{\mathcal{L}}}{\partial\dot{\tilde{\psi}}(\vec{x},t)} = i\tilde{\psi}^\dagger(\vec{x},t) \end{aligned} \tag{3.54}$$

This explains why we chose the sign of the first term in $\tilde{\mathcal{L}}$ to be different from that of $\mathcal{L}$. With this choice, the variables and the conjugate momenta become identical for the tilde and the nontilde systems. Correspondingly, they satisfy similar canonical commutation relations (as is assumed in thermofield dynamics)

$$\begin{aligned} [\psi(\vec{x},t),\psi^\dagger(\vec{y},t)] &= \delta^3(x-y) \\ [\psi(\vec{x},t),\psi(\vec{y},t)] &= 0 = [\psi^\dagger(\vec{x},t),\psi^\dagger(\vec{y},t)] \end{aligned}$$

$$
\begin{aligned}
[\tilde{\psi}(\vec{x},t),\tilde{\psi}^{\dagger}(\vec{y},t)] &= \delta^3(x-y) \\
[\tilde{\psi}(\vec{x},t),\tilde{\psi}(\vec{y},t)] &= 0 = [\tilde{\psi}^{\dagger}(\vec{x},t),\tilde{\psi}^{\dagger}(\vec{y},t)]
\end{aligned} \tag{3.55}
$$

and the two systems become identical. We assume further that the tilde fields commute with the nontilde ones. This brings out one of the rules for tilde "conjugation", namely, the Lagrangian for the tilde system is obtained from that of the original system by replacing the field variables by the tilde ones and complex conjugating all the coefficients in the Lagrangian density. This is also commonly represented by saying that

$$
\widetilde{c\psi} = c^{*}\tilde{\psi} \tag{3.56}
$$

This is quite crucial, even when the Lagrangian density does not involve any complex parameters, in bringing out the correct analytic structure of the finite temperature Greens functions. We would like to emphasize here that the tilde conjugation rules are to be taken as a postulate within the framework of thermofield dynamics.

The Hamiltonian densities for the tilde and the nontilde systems are given by

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{2m}\vec{\nabla}\psi^{\dagger}(\vec{x},t)\cdot\vec{\nabla}\psi(\vec{x},t) \\
\tilde{\mathcal{H}} &= \frac{1}{2m}\vec{\nabla}\tilde{\psi}^{\dagger}(\vec{x},t)\cdot\vec{\nabla}\tilde{\psi}(\vec{x},t)
\end{aligned} \tag{3.57}
$$

leading to the total Hamiltonian for the combined system to be

$$
\hat{H} = H - \tilde{H} \tag{3.58}
$$

It is straightforward to check using eq. (3.55) that the dynamical equations for the system can be written as Heisenberg equations of the form

$$\begin{aligned} i\dot{\psi}(\vec{x},t) &= [\psi(\vec{x},t),\hat{H}] = -\frac{1}{2m}\vec{\nabla}^2\psi(\vec{x},t) \\ i\dot{\tilde{\psi}}(\vec{x},t) &= [\tilde{\psi}(\vec{x},t),\hat{H}] = \frac{1}{2m}\vec{\nabla}^2\tilde{\psi}(\vec{x},t) \end{aligned} \tag{3.59}$$

This shows that $\hat{H}$ can, indeed, be thought of as the generator of time evolution for the combined system.

Let us next expand the Schrödinger fields in a plane wave basis.

$$\begin{aligned} \psi(\vec{x},t) &= \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}\psi_{\vec{k}}(t) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{-i\omega_k t + i\vec{k}\cdot\vec{x}}a_{\vec{k}} \\ \tilde{\psi}(\vec{x},t) &= \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}\tilde{\psi}_{\vec{k}}(t) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}}\tilde{a}_{\vec{k}} \end{aligned} \tag{3.60}$$

There are two things to note here. First, we have quantized in a finite, cubic box of volume V, for simplicity, which leads to discrete values for $\vec{k}$. The energy is given by

$$\omega_k = \frac{\vec{k}^2}{2m} \tag{3.61}$$

The expansion for $\tilde{\psi}$ can be obtained from that of ψ using the tilde conjugation rule of eq. (3.56). Moreover, we can think of $a_{\vec{k}}$ and $\tilde{a}_{\vec{k}}$ as representing the annihilation operators for the original system and the tilde system respectively. But from the structure of the exponential factor multiplying the two, it is quite clear that $\tilde{a}_{\vec{k}}$ is more like a creation

operator as we had noted earlier. In terms of these operators, we can write down the Hamiltonian and it takes the form

$$\begin{aligned} H &= \sum_{\vec{k}} \omega_k a^\dagger_{\vec{k}} a_{\vec{k}} \\ \tilde{H} &= \sum_{\vec{k}} \omega_k \tilde{a}^\dagger_{\vec{k}} \tilde{a}_{\vec{k}} \\ \hat{H} &= H - \tilde{H} = \sum_{\vec{k}} \omega_k \left(a^\dagger_{\vec{k}} a_{\vec{k}} - \tilde{a}^\dagger_{\vec{k}} \tilde{a}_{\vec{k}} \right) \end{aligned} \tag{3.62}$$

The Hilbert space for the combined system can now be constructed in a straightforward manner. Let us simply note here, following the discussion of the last section, that we can now define the thermal vacuum as

$$|0,\beta\rangle = Z^{-1/2}(\beta) \sum_{\vec{k}} \sum_{n_{\vec{k}}} \mathrm{e}^{-\beta\omega_k n_{\vec{k}}/2} \, |n_{\vec{k}}, \tilde{n}_{\vec{k}}\rangle \tag{3.63}$$

where the partition function for the system is given by

$$Z(\beta) = \sum_{\vec{k}} \frac{1}{1 - \mathrm{e}^{-\beta\omega_k}} \tag{3.64}$$

We also note following the discussion of eq. (3.41) that we can define a generator of Bogoliubov transformation as (see eqs. (3.39) and (3.40))

$$Q(\theta) = -i \sum_{\vec{k}} \theta_k(\beta) \left(\tilde{a}_{\vec{k}} a_{\vec{k}} - a^\dagger_{\vec{k}} \tilde{a}^\dagger_{\vec{k}} \right) \tag{3.65}$$

which would lead to a formally unitary operator

$$U(\theta) = \mathrm{e}^{-iQ(\theta)} = \mathrm{e}^{-\sum_{\vec{k}} \theta_k(\beta) \left(\tilde{a}_{\vec{k}} a_{\vec{k}} - a^\dagger_{\vec{k}} \tilde{a}^\dagger_{\vec{k}} \right)} \tag{3.66}$$

This unitary operator will connect the thermal vacuum to the usual one in the standard manner, namely,

$$|0,\beta\rangle = U(\theta)|0,\beta\rangle \tag{3.67}$$

provided we identify (see eq. (3.42))

$$\cosh\theta_k(\beta) = \frac{1}{\sqrt{1-\mathrm{e}^{-\beta\omega_k}}} \qquad \sinh\theta_k(\beta) = \frac{\mathrm{e}^{-\beta\omega_k/2}}{\sqrt{1-\mathrm{e}^{-\beta\omega_k}}} \tag{3.68}$$

Given these, the thermal operators can also be calculated in a straightforward manner (see eq. (3.44)) and we have

$$\begin{aligned}\begin{pmatrix} a_{\vec{k}}(\beta) \\ \tilde{a}^\dagger_{\vec{k}}(\beta) \end{pmatrix} &= \overline{U}(\theta_k) \begin{pmatrix} a_{\vec{k}} \\ \tilde{a}^\dagger_{\vec{k}} \end{pmatrix} \\ &= \begin{pmatrix} \cosh\theta_k(\beta) & -\sinh\theta_k(\beta) \\ -\sinh\theta_k(\beta) & \cosh\theta_k(\beta) \end{pmatrix} \begin{pmatrix} a_{\vec{k}} \\ \tilde{a}^\dagger_{\vec{k}} \end{pmatrix}\end{aligned} \tag{3.69}$$

Alternately, we can write this in terms of the field variables as

$$\begin{aligned}\begin{pmatrix} \psi_{\vec{k}}(t,\beta) \\ \tilde{\psi}^\dagger_{\vec{k}}(t,\beta) \end{pmatrix} &= \overline{U}(\theta_k) \begin{pmatrix} \psi_{\vec{k}}(t) \\ \tilde{\psi}^\dagger_{\vec{k}} \end{pmatrix} \\ &= \begin{pmatrix} \cosh\theta_k(\beta) & -\sinh\theta_k(\beta) \\ -\sinh\theta_k(\beta) & \cosh\theta_k(\beta) \end{pmatrix} \begin{pmatrix} \psi_{\vec{k}}(t) \\ \tilde{\psi}^\dagger_{\vec{k}}(t) \end{pmatrix}\end{aligned} \tag{3.70}$$

With these relations, it is easy to see that the thermal vacuum satisfies as in eq. (3.46)

$$\begin{aligned} a_{\vec{k}}(\beta)|0,\beta)\rangle &= \left(a_{\vec{k}}\cosh\theta_k(\beta) - \tilde{a}^\dagger_{\vec{k}}\sinh\theta_k(\beta)\right)|0,\beta\rangle = 0 \\ \tilde{a}_{\vec{k}}(\beta)|0,\beta\rangle &= \left(\tilde{a}_{\vec{k}}\cosh\theta_k(\beta) - a^\dagger_{\vec{k}}\sinh\theta_k(\beta)\right)|0,\beta\rangle = 0 \end{aligned} \tag{3.71}$$

Once again, we see that the thermal vacuum is truly annihilated by the thermal annihilation operators. This also brings out the fact that a quantum of the tilde particle behaves like a hole of the original system and that the thermal vacuum contains an equal number of particles and holes which follows from

$$\begin{aligned} \widehat{N}|0,\beta\rangle &= 0 \\ \widehat{H}|0,\beta\rangle &= 0 \end{aligned} \tag{3.72}$$

Let us note here that a Bogoliubov transformation is normally associated with a spontaneously broken symmetry. It is interesting, therefore, to ask what is the symmetry that is spontaneously broken at finite temperature. To see that, let us note that the Hamiltonian in eq. (3.58) or (3.62) is invariant under the transformations

$$\begin{aligned} a_{\vec{k}} &\rightarrow a_{\vec{k}} \cosh\alpha_{\vec{k}} - \tilde{a}^{\dagger}_{\vec{k}} \sinh\alpha_{\vec{k}} \\ \tilde{a}_{\vec{k}} &\rightarrow \tilde{a}_{\vec{k}} \cosh\alpha_{\vec{k}} - a^{\dagger}_{\vec{k}} \sinh\alpha_{\vec{k}} \end{aligned} \tag{3.73}$$

where $\alpha_{\vec{k}}$ is an arbitrary global parameter. This is a symmetry that is not present in the original system – rather it develops for the combined system with the tilde fields. It can be easily checked, with the standard commutation relations, that the generator of the infinitesimal form of this symmetry transformations is

$$Q(\alpha) = -i \sum_{\vec{k}} \alpha_{\vec{k}} \left(\tilde{a}_{\vec{k}} a_{\vec{k}} - a^{\dagger}_{\vec{k}} \tilde{a}^{\dagger}_{\vec{k}} \right)$$

Furthermore, it is quite clear from the structure of the thermal vacuum in eq. (3.63) that

$$Q(\alpha)|0,\beta\rangle \neq 0 \tag{3.74}$$

The generator of the symmetry does not annihilate the thermal vacuum. (In fact, if it did, the thermal vacuum will coincide with the vacuum of the original theory with doubled degrees of freedom.) In other words, the symmetry between the original and the tilde system that is present in the Hamiltonian at finite temperature is spontaneously broken by the thermal vacuum. However, we do not expect a massless, Goldstone boson in this case because at finite temperature Lorentz invariance is not manifest and the spectrum of $\widehat{H}$ is indefinite (non-positive).

3.5 Free Klein-Gordon Theory

As a second example of field theories, let us look at the free Klein-Gordon theory of real fields at finite temperature. We continue to work in $3+1$ dimensions and the Lagrangian density is the standard one, namely,

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 \tag{3.75}$$

The Lagrangian density for the tilde system can now be written down in a straightforward manner

$$\tilde{\mathcal{L}} = \frac{1}{2}\,\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \frac{m^2}{2}\,\tilde{\phi}^2 \tag{3.76}$$

so that the complete Lagrangian for the combined system can be written as

$$\widehat{\mathcal{L}} = \mathcal{L} - \widetilde{\mathcal{L}}$$

We note here that because the fields are real in this case, the tilde conjugation rule does not seem to have any apparent effect. That is not true, however, if we remember that the Feynman boundary condition is equivalent to adding an infinitesimal, imaginary term to the quadratic part of the Lagrangian density. The sign of that term would change under complex conjugation implying that the $i\epsilon$–prescription will be different for the tilde fields compared to the original system.

We can now go on and quantize the system, construct the combined Hilbert space as well as the thermal vacuum. However, let us, instead, ask how one can construct the thermal propagator of the theory from all the information we have. To that end, let us introduce the doublet (From our earlier discussion in eq. (3.30) or (3.43) or (3.70), we recognize that we should group ϕ and $\widetilde{\phi}^\dagger$ into a doublet. In this case, however, the fields are real.)

$$\Phi = \begin{pmatrix} \phi \\ \widetilde{\phi} \end{pmatrix} \tag{3.77}$$

Then, we can write the propagator for the combined system at zero temperature to be

$$iG(x-y) = \langle 0,\widetilde{0}|T(\Phi(x)\Phi(y))|0,\widetilde{0}\rangle \tag{3.78}$$

We should recognize that G, in the present case, is a 2×2 matrix whose Fourier transform has the form (note the Feynman prescription for the tilde field and compare with our earlier discussion)

$$G(k) = \begin{pmatrix} \frac{1}{k^2 - m^2 + i\epsilon} & 0 \\ 0 & -\frac{1}{k^2 - m^2 - i\epsilon} \end{pmatrix} \tag{3.79}$$

We can now determine the thermal propagator quite easily from the following observation. By definition

$$\begin{aligned} iG_\beta(x-y) &= \langle 0,\beta|T(\Phi(x)\Phi(y))|0,\beta\rangle \\ &= \langle 0,\tilde{0}|T(U^\dagger(\theta)\Phi(x)\Phi(y)U(\theta))|0,\tilde{0}\rangle \\ &= \langle 0,\tilde{0}|T(U(-\theta)\Phi(x)U^\dagger(-\theta)U(-\theta)\Phi(y)U^\dagger(-\theta))|0,\tilde{0}\rangle \\ &= \langle 0,\tilde{0}|T(\overline{U}(-\theta)\Phi(x)\Phi(y)\overline{U}^T(-\theta))|0,\tilde{0}\rangle \\ &= \overline{U}(-\theta)\langle 0,\tilde{0}|T(\Phi(x)\Phi(y))|0,\tilde{0}\rangle\overline{U}^T(-\theta) \\ &= \overline{U}(-\theta)iG(x-y)\overline{U}^T(-\theta) \end{aligned} \tag{3.80}$$

Here we have used our earlier observations in eqs. (3.41) and (3.44). From eqs. (3.79) and (3.80), we can now immediately obtain the thermal propagator in the formalism of thermofield dynamics to be

$$\begin{aligned} iG_\beta(k) &= \overline{U}(-\theta)\, iG(k)\, \overline{U}^T(-\theta) \\ &= \begin{pmatrix} \cosh\theta_k(\beta) & \sinh\theta_k(\beta) \\ \sinh\theta_k(\beta) & \cosh\theta_k(\beta) \end{pmatrix} \begin{pmatrix} \frac{i}{k^2 - m^2 + i\epsilon} & 0 \\ 0 & -\frac{i}{k^2 - m^2 - i\epsilon} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \cosh\theta_k(\beta) & \sinh\theta_k(\beta) \\ \sinh\theta_k(\beta) & \cosh\theta_k(\beta) \end{pmatrix} \end{aligned} \tag{3.81}$$

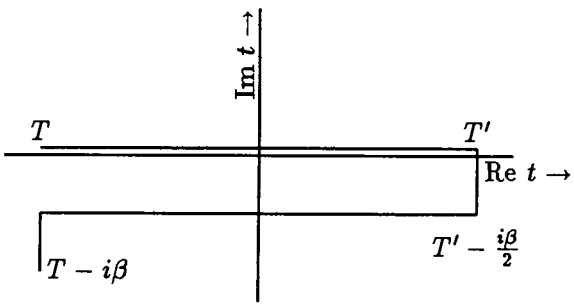
Using the standard form for $\theta_k(\beta)$ for bosonic theories (see eq. (3.68)), we can now simplify and write the finite temperature Greens functions

for the free Klein-Gordon theory in thermofield dynamics to be

$$G_\beta(k) = \begin{pmatrix} \frac{1}{k^2-m^2+i\epsilon} & 0 \\ 0 & -\frac{1}{k^2-m^2-i\epsilon} \end{pmatrix}$$

$$-2i\pi n_B(|k^0|)\delta(k^2-m^2)\begin{pmatrix} 1 & e^{\beta|k^0|/2} \\ e^{\beta|k^0|/2} & 1 \end{pmatrix} \qquad (3.82)$$

There are several things to note here. First, the finite temperature Greens functions (propagators) separate into a sum of two terms – one corresponding to zero temperature and the second explicitly dependent on temperature. This is very much like the case of the propagator in the closed time path formalism which we had discussed earlier. Second, the propagator has a 2×2 matrix structure very much like the closed time path formalism. However, the structure of the propagator in thermofield dynamics is not the same as that in eq. (2.36). In fact, we note that while the diagonal terms in the propagator matrices in eqs. (2.36) and (3.82) coincide, the off-diagonal terms are different because of the $e^{\beta|k^0|/2}$ factor. On the other hand, we note that the matrix propagator in eq. (3.82) is identical with that of eqs. (2.46) and (2.47) for $\sigma = \frac{\beta}{2}$ of the generalized closed time path. This suggests that we can describe thermofield dynamics in a path integral formulation as well where the time contour in the complex time plane has the following form.



One can identify the tilde fields as living on the second real branch of this contour and this is how the matrix structure for the propagator arises. Note that in the context of thermofield dynamics, the elements of the propagator matrix do not satisfy any natural relations like in eq. (2.36). Furthermore, even though we can think of the tilde field as the original field on the second branch of the time contour, the physical meanings are quite different from the ones in the closed time path formalism. In thermofield dynamics the tilde fields are supposed to commute (anti-commute) with the original field which is different from what one would expect the commutation relations on the two real branches to be in the closed time path formalism. Thus, while there are similarities between the two formalisms, one should not push the analogies too far. Finally, let us note here that if there are interactions present, we can now introduce the corresponding interactions involving the tilde fields as well through the tilde conjugation relations. The complete interacting Lagrangian density, then, gives rise to the interaction vertices where the interaction vertices involving the tilde fields (in general) will have a relative negative sign compared to those for the original system. One can use the matrix form of the propagator in eq. (3.82) and carry out

perturbative calculations in a standard manner. As we discussed earlier, the doubling of the fields, among other things, leads to cancellation of ill-defined terms in the loop calculations and the zero temperature counter terms are sufficient to renormalize the theory.

3.6 KMS Condition

As we have seen in the first chapter, the KMS conditions are at the heart of equilibrium statistical mechanics and arise from the fact that we are evaluating a trace when we are calculating an ensemble average. In thermofield dynamics, the KMS condition also follows quite easily from the thermal condition (see eq. (3.33) or (3.46)). We note (We are going to suppress the spatial coordinates for simplicity.) that the thermal condition says (The second relation is obtained from the adjoint of the second relation in eq. (3.50).)

$$\begin{aligned} A(t)|0,\beta\rangle &= e^{\beta\widehat{H}/2}\widetilde{A}^{\dagger}(t)|0,\beta\rangle \\ \langle 0,\beta|A(t) &= (-1)^{|A|}\langle 0,\beta|\widetilde{A}^{\dagger}(t)e^{\beta\widehat{H}/2} \end{aligned} \tag{3.83}$$

Let us use this to calculate the ensemble average of the correlation of two arbitrary operators.

$$\begin{aligned} \langle 0,\beta|A(t)B(t')|0,\beta\rangle &= (-1)^{|A|}\langle 0,\beta|\widetilde{A}^{\dagger}(t)e^{\beta\widehat{H}/2}B(t')|0,\beta\rangle \\ &= (-1)^{|A|}\langle 0,\beta|\widetilde{A}^{\dagger}(t+i\beta/2)B(t')|0,\beta\rangle \\ &= (-1)^{|A|(1+|B|)}\langle 0,\beta|B(t')\widetilde{A}^{\dagger}(t+i\beta/2)|0,\beta\rangle \\ &= (-1)^{|A|(1+|B|)}\langle 0,\beta|B(t')e^{-\frac{\beta\widehat{H}}{2}}A(t+i\beta/2)|0,\beta\rangle \\ &= \langle 0,\beta|B(t')A(t+i\beta)|0,\beta\rangle \end{aligned} \tag{3.84}$$

This is the KMS condition and to obtain this, we have used the thermal condition in eq. (3.72) as well as in eq. (3.83). In addition, we have used, in the last step, the fact that the thermal vacuum expectation value vanishes if both the operators do not have the same Grassmann parity. It is clear that eq. (3.84) holds for bosonic as well as fermionic operators.

3.7 Dispersion Relations

In chapter 2, we derived the dispersion relations for the free, bosonic Greens functions (propagators). Here we will derive it for arbitrary operators. Let us consider two arbitrary operators, $A(t)$ and $B(t)$, both of which are assumed to be either bosonic or fermionic. For simplicity, we are suppressing the spatial coordinates. Let $I_{AB}(p^0)$ represent the Fourier transform of the thermal correlation function of $A(t)$ and $B(t')$, i.e.,

$$\langle 0,\beta|A(t)B(t')|0,\beta\rangle = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(t-t')}\, I_{AB}(p^0) \tag{3.85}$$

Once again, we are suppressing the dependence of the operators on other coordinates. If we now use the KMS condition in eq. (3.84), namely,

$$\langle 0,\beta|A(t)B(t')|0,\beta\rangle = \langle 0,\beta|B(t')A(t+i\beta)|0,\beta\rangle$$

then, we can write

$$\begin{aligned}\langle 0,\beta|B(t')A(t)|0,\beta\rangle &= \langle 0,\beta|A(t-i\beta)B(t')|0,\beta\rangle \\ &= \int_{-\infty}^{\infty} \frac{dp^0}{2\pi}\, \mathrm{e}^{-ip^0(t-t')}\, \mathrm{e}^{-\beta p^0}\, I_{AB}(p^0)\end{aligned} \tag{3.86}$$

Given these, we can now calculate the retarded correlation functions of the two operators from the standard definition (see eq. (2.39))

$$iR_{AB}(t-t') = \theta(t-t')\,\langle 0,\beta|[A(t),B(t')]_{\mp}|0,\beta\rangle \tag{3.87}$$

Here the subscript $\mp$ refers to the cases where both A and B are bosonic or fermionic. If we define the Fourier transform as

$$R_{AB}(t-t') = \int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\,\mathrm{e}^{-ip^0(t-t')}\,R_{AB}(p^0) \tag{3.88}$$

and use a standard integral representation for the θ–function, namely,

$$\theta(t-t') = \int_{-\infty}^{\infty}\frac{dp^0}{2i\pi}\,\frac{\mathrm{e}^{ip^0(t-t')}}{p^0-i\epsilon} \tag{3.89}$$

then, it is easy to show from eqs. (3.85–3.89) that

$$\begin{aligned}
&R_{AB}(t-t') \\
&= (-i)\int_{-\infty}^{\infty}\frac{dp^0}{2i\pi}\int_{-\infty}^{\infty}\frac{dp'^0}{2\pi}\,\frac{\mathrm{e}^{-i(p'^0-p^0)(t-t')}}{p^0-i\epsilon}\,(1\mp\mathrm{e}^{-\beta p'^0})\,I_{AB}(p'^0) \\
&= (-i)\int_{-\infty}^{\infty}\frac{dp^0}{2i\pi}\int_{-\infty}^{\infty}\frac{dp'^0}{2\pi}\,\frac{\mathrm{e}^{-ip^0(t-t')}}{p'^0-p^0-i\epsilon}\,(1\mp\mathrm{e}^{-\beta p'^0})\,I_{AB}(p'^0) \\
&= \int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\,\mathrm{e}^{-ip^0(t-t')}\int_{-\infty}^{\infty}\frac{dp'^0}{2\pi}\,\frac{I_{AB}(p'^0)}{p^0-p'^0+i\epsilon}\,(1\mp\mathrm{e}^{-\beta p'^0})
\end{aligned} \tag{3.90}$$

Comparing this with eq. (3.88), we obtain

$$R_{AB}(p^0) = \int_{-\infty}^{\infty}\frac{dp'^0}{2\pi}\,\frac{I_{AB}(p'^0)}{p^0-p'^0+i\epsilon}\,(1\mp\mathrm{e}^{-\beta p'^0}) \tag{3.91}$$

Similarly, the causal correlation functions can also be calculated in a straightforward manner from the definitions

$$\begin{aligned} iG_{AB}(t-t') &= \langle 0,\beta|T(A(t)B(t'))|0,\beta\rangle \\ G_{AB}(t-t') &= \int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\,\mathrm{e}^{-ip^0(t-t')}\,G_{AB}(p^0) \end{aligned} \tag{3.92}$$

where the time ordering is defined to be

$$T(A(t)B(t')) = \theta(t-t')\,A(t)B(t') \pm \theta(t'-t)\,B(t')A(t) \tag{3.93}$$

with + for bosonic operators and − for the fermionic ones. Using eqs. (3.85), (3.86) and (3.89) we can now obtain

$$\begin{aligned} G_{AB}(t-t') &= \int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\,\mathrm{e}^{-ip^0(t-t')}\int_{-\infty}^{\infty}\frac{dp'^0}{2\pi}\,I_{AB}(p'^0) \\ &\quad\times\left(\frac{1}{p^0-p'^0+i\epsilon}\mp\frac{\mathrm{e}^{-\beta p'^0}}{p^0-p'^0-i\epsilon}\right) \end{aligned} \tag{3.94}$$

From eq. (3.92), we, then, obtain

$$G_{AB}(p^0) = \int_{-\infty}^{\infty}\frac{dp'^0}{2\pi}\,I_{AB}(p'^0)\left(\frac{1}{p^0-p'^0+i\epsilon}\mp\frac{\mathrm{e}^{-\beta p'^0}}{p^0-p'^0-i\epsilon}\right) \tag{3.95}$$

It is easy to see from eqs. (3.91) and (3.95) that

$$\begin{aligned} \mathrm{Re}\,R_{AB}(p^0) &= \mathrm{Re}\,G_{AB}(p^0) \\ \mathrm{Im}\,R_{AB}(p^0) &= \begin{cases} \tanh(\beta p^0/2)\,\mathrm{Im}\,G_{AB}(p^0) & \text{for bosons} \\ \coth(\beta p^0/2)\,\mathrm{Im}\,G_{AB}(p^0) & \text{for fermions} \end{cases} \end{aligned} \tag{3.96}$$

This is the generalization of the dispersion relations obtained in eq. (2.51) for arbitrary operators. We note here (as pointed out in the last chapter) that the derivation of these relations is general enough that

they are true even in an interacting theory. By choosing the operators A and B to be fundamental field variables, we can obtain conditions on the spectral function from these relations. Furthermore, the LSZ reduction of zero temperature can now be extended in a standard manner to nonzero temperatures.

3.8 Goldstone Theorem

The thermal field theories are particularly interesting from the point of view of the study of spontaneous symmetry breaking and restoration of symmetries. However, a naive application of the ideas of symmetry breaking into finite temperature runs into difficulties. This can be seen as follows. Let H denote the Hamiltonian governing the dynamics of the system and let U denote a unitary symmetry of the dynamical system. Then, by definition,

$$UHU^{\dagger} = H$$

If we write the unitary operator in terms of the generator of the infinitesimal symmetry, Q, then it takes the form (α is the constant parameter of transformation.)

$$U = \mathrm{e}^{-i\alpha Q} \tag{3.97}$$

In such a case, the invariance of the Hamiltonian can be written as

$$[Q, H] = 0 \tag{3.98}$$

We say that a symmetry is spontaneously broken if the vacuum is not invariant under the symmetry transformations even when the Hamiltonian for the system is. Explicitly, if

$$Q|0\rangle \neq 0 \tag{3.99}$$

then, we say that the symmetry is spontaneously broken. An alternate way to characterize spontaneous symmetry breaking is to note that in such a case, there always exists an operator A such that (We will discuss this in more detail in chapter 6.)

$$\langle 0|[Q, A]|0\rangle \neq 0 \tag{3.100}$$

The field A is, of course, what is conventionally known as the Goldstone field which, for manifestly Lorentz invariant theories with a positive semi-definite Hilbert space, is known to give rise to massless quanta.

At finite temperature, on the other hand, we have to work with ensemble averages. We note that for any arbitrary operator A, we can conclude from eq. (3.98) that

$$\langle 0, \beta|[Q, A]|0, \beta\rangle = \mathrm{Tr}\, \mathrm{e}^{-\beta H}\,[Q, A] = 0 \tag{3.101}$$

which follows simply from the cyclicity of the trace operation. This would, therefore, seem to suggest that at finite temperature, there cannot be any spontaneous symmetry breaking since eq. (3.101) must hold for any arbitrary operator A. The flaw in this argument, of course, lies in the fact that for infinite dimensional operators, cyclicity of trace does not necessarily hold as we had also noted in chapter 1. The simplest example of this arises in quantum mechanics where

$$[x, p] = i\hbar$$

which, if we were to use the cyclicity of trace, leads to the inconsistent result that

$$\mathrm{Tr}\,[x, p] = 0 \qquad \text{where as} \qquad \mathrm{Tr}\, i\hbar \neq 0$$

In addition, the cyclicity property may also be violated if we have singular operators. Thus, one has to be careful in the study of spontaneous symmetry breaking at finite temperature.

As in zero temperature, there are only two possibilities even at finite temperature.

1. For all possible operators, A, we have

$$\langle 0, \beta | [Q, A] | 0, \beta \rangle = 0$$

In this case, we say that the symmetry is a true symmetry of the system and that

$$Q|0, \beta\rangle = 0$$

2. There exists an operator A for which

$$\langle 0, \beta | [Q, A] | 0, \beta \rangle \neq 0 \tag{3.102}$$

In such a case, we say that the symmetry is spontaneously broken and that the thermal vacuum is not annihilated by the symmetry generator Q. To see the consequences of symmetry breaking, let us note the

following. If the symmetry is a continuous global symmetry, then when matter is sufficiently localized and if there are no long range forces, we have a continuity equation following from Nöether's theorem

$$\partial_\mu J^\mu(\vec{x},t) = 0$$

The generator of the infinitesimal symmetry is related to the conserved current through the relation

$$Q = \int d^3x\, J^0(\vec{x},t) \tag{3.103}$$

and we note that the charge, Q, is, in fact, independent of time which follows from the continuity equation.

Given these, we note now, following eqs. (3.85)-(3.86) that

$$\langle 0,\beta|[J^0(y),A(x)]|0,\beta\rangle = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(y-x)}\, I(p^0,\vec{p})(1-e^{-\beta p^0}) \tag{3.104}$$

Upon integrating over d^3y (see eq. (3.103)), we obtain

$$\langle 0,\beta|[Q,A]|0,\beta\rangle = \int \frac{dp^0}{2\pi}\, e^{-ip^0(y^0-x^0)}\, I(p^0,0)(1-e^{-\beta p^0}) \tag{3.105}$$

If, for some operator A, eq. (3.102) holds, then, the right hand side of eq. (3.105) must also be nonzero. Let us also note that the left hand side of eq. (3.105) is independent of y^0 since the charge Q is a constant. Thus, differentiating eq. (3.105) with respect to y^0, we obtain

$$\int \frac{dp^0}{2\pi}\, e^{-ip^0(y^0-x^0)}\, p^0 I(p^0,0)(1-e^{-\beta p^0}) = 0 \tag{3.106}$$

This implies that, for eq. (3.105) to be nonzero, we must have

$$I(p^0, 0)(1 - e^{-\beta p^0}) = a\,\delta(p^0) \tag{3.107}$$

where a is a constant. This shows that, in this case, certain correlation functions will develop a delta function singularity.

Thus, we can state the Goldstone theorem at finite temperature as follows. If in the absence of long range forces, a theory with sufficiently localized fields has a spontaneous breakdown of a continuous global symmetry, then, there must appear quanta with zero energy. These quanta need not be massless since we are no longer in a covariant framework at finite temperature. (In fact, as we have seen before, mass is not a uniquely defined concept at finite temperature.) We can think of these quanta with zero energy as quasi-particle states. In fact, a little analysis, following from our earlier discussions, shows that the Goldstone states can be thought of as composites of tilde and nontilde states.

3.9 References

Kubo, R, J. Phys. Soc. Japan **12** (1957) 570.

Landsman, N.P and **Ch. G. Van Weert**, Phys. Reports **145** (1987) 141.

Martin, P.C and **Schwinger, J**, Phys. Rev. **115** (1959) 1342.

Niemi, A and **Semenoff, G**, Ann. Phys. **152** (1984) 105.

Takahashi, Y and **Umezawa, H**, Collective Phenomena **2** (1975) 55.

Umezawa, H, H. Matsumoto and **M. Tachiki**, "Thermofield Dynamics and Condensed States", North-Holland (1982).

Chapter 4

Gauge Theories

Gauge theories are theories with local invariances. In the simplest form, they might just describe theories invariant under local phase transformations of the field variables, but can also have more complicated symmetry structures. As we understand now, gauge theories are absolutely essential to describe the fundamental forces of nature. It is, therefore, crucial to understand the structure of such theories, both at zero temperature as well as at finite temperature. In the next few sections, we will first try to understand the essential features of these theories at zero temperature before going onto the case of finite temperature.

4.1 Gauge Theories at Zero Temperature

The simplest of the gauge theories, of course, dates back to Maxwell. Maxwell's theory, as a quantum theory, is based on a local $U(1)$ phase invariance of the matter fields. Since the symmetry group, $U(1)$, has only one generator, the symmetry algebra is Abelian and correspondingly, Maxwell's theory describes an Abelian gauge theory. This theory describes quite well the interactions between electrons and photons. However, being an Abelian gauge theory, the structure of this theory – Quantum Electrodynamics (QED) – is much simpler compared to the other gauge theories that one deals with. To bring out various interesting structures associated with gauge theories, therefore, we would confine ourselves to non-Abelian gauge theories or Yang-Mills theories in this section.

Let us consider a simple group such as $SU(n)$. This is a Lie group with n^2-1 Hermitian generators, T^a, $a = 1, 2, \cdots, n^2-1$, satisfying the Lie algebra (Summation over repeated indices is assumed throughout.)

$$[T^a, T^b] = if^{abc}\, T^c \tag{4.1}$$

where $a, b, c = 1, 2, \cdots, n^2-1$ and the constants, f^{abc}, which are completely antisymmetric in the three indices, are known as the structure constants. We note that unlike the group $U(1)$, here the generators of the Lie algebra do not commute and consequently, such an algebra is known as a non-Abelian Lie algebra. The structure constants (including

the factor of i), in fact, provide a representation of the generators of the Lie algebra known as the adjoint representation.

The simplest way to introduce the Yang-Mills theory is to look at a free fermion theory where there are n fermions belonging to the fundamental representation of the Lie group $SU(n)$. Thus, we can write the Lagrangian density for such a free theory as

$$\mathcal{L}_0 = \overline{\psi}^i(x)(i\gamma^\mu\partial_\mu - m)\psi^i(x) \tag{4.2}$$

where we assume the fermions to have a mass m and $i = 1, 2, \cdots, n$. It is easy to check that this Lagrangian density is invariant under an infinitesimal global $SU(n)$ transformation of the form

$$\begin{aligned} \delta\psi^i &= i\epsilon^a(t^a)^{ij}\psi^j \\ \delta\overline{\psi}^i &= -i\epsilon^a\overline{\psi}^j(t^a)^{ji} \end{aligned} \tag{4.3}$$

where the infinitesimal parameters of the transformation, ϵ^a, are assumed to be space-time independent and that t^a's represent the generators of $SU(n)$ defined in eq. (4.1) in the fundamental representation. (We will reserve T^a to represent the generators in the adjoint representation. We also note here that the global invariance of the Lagrangian density in eq. (4.2) is really $U(n)$. But, for simplicity, we will simply work with the $SU(n)$ part of the symmetry of the free theory.)

Just like in the case of Quantum Electrodynamics, we can now ask when this global symmetry can be promoted to a local symmetry. We note that if the parameters of the transformation in eq.

(4.3) were local, namely, if $\epsilon^a = \epsilon^a(x)$, then the Lagrangian density in eq. (4.2) will no longer be invariant under the transformations in eq. (4.3). Rather, the change in the Lagrangian density will have the form

$$\begin{aligned} \delta\mathcal{L}_0 &= \delta\overline{\psi}^i(i\gamma^\mu\partial_\mu - m)\psi^i + \overline{\psi}^i(i\gamma^\mu\partial_\mu - m)\delta\psi^i \\ &= -(\partial_\mu\epsilon^a(x))\,\overline{\psi}^i\gamma^\mu(t^a)^{ij}\psi^j \end{aligned} \tag{4.4}$$

We also note that the $SU(n)$ current (Nöether current) associated with the global invariance of the free theory has the form

$$J^{\mu a} = \overline{\psi}^i\gamma^\mu(t^a)^{ij}\psi^j \tag{4.5}$$

Therefore, if we were to introduce a gauge interaction, completely parallel with QED, it will have the form

$$\mathcal{L}_{int} = J^{\mu a}A^a_\mu = \overline{\psi}^i\gamma^\mu(t^a)^{ij}\psi^j A^a_\mu \tag{4.6}$$

We can think of the A^a_μ as the appropriate gauge fields for the present case and we note that, unlike the case of QED, here there will be $a = 1, 2, \cdots, n^2 - 1$ gauge fields. The change in the interaction Lagrangian density, eq. (4.6), under a local transformation of the form eq. (4.3) can be easily calculated.

$$\begin{aligned} \delta\mathcal{L}_{int} &= \delta\overline{\psi}^i\gamma^\mu(t^a)^{ij}\psi^j A^a_\mu + \overline{\psi}^i\gamma^\mu(t^a)^{ij}\delta\psi^j A^a_\mu + \overline{\psi}^i\gamma^\mu(t^a)^{ij}\psi^j\delta A^a_\mu \\ &= -f^{abc}A^b_\mu\epsilon^c\overline{\psi}^i\gamma^\mu(t^a)^{ij}\psi^j + \overline{\psi}^i\gamma^{mu}(t^a)^{ij}\psi^j\delta A^a_\mu \end{aligned} \tag{4.7}$$

In deriving the last line of the above result, we have used the commutation relations for the generators, t^a, given in eq. (4.1).

Thus, we see from eqs. (4.4) and (4.7) that the total Lagrangian density, including the interactions,

$$\mathcal{L}_{TOT} = \mathcal{L}_0 + \mathcal{L}_{int}$$

changes under the transformations of eq. (4.3) (when the parameters are local) as

$$\delta\mathcal{L}_{TOT} = [\delta A^a_\mu - (\partial_\mu \epsilon^a + f^{abc} A^b_\mu \epsilon^c)]\overline{\psi}^i \gamma^\mu (t^a)^{ij} \psi^j \tag{4.8}$$

It is clear now that the total Lagrangian density will be invariant under local transformations of the form eq. (4.3) if we choose the gauge fields to transform also as

$$\delta A^a_\mu(x) = (\partial_\mu \epsilon^a + f^{abc} A^b_\mu \epsilon^c) \tag{4.9}$$

We note that this is precisely how the gauge field transforms in QED for local gauge invariance where, for $U(1)$, the structure constants vanish.

For any field, belonging to a representation of $SU(n)$, we can define a covariant derivative with the generators of the group, in that particular representation, as well as the gauge fields (connections). Thus, for example, for the fermions in the fundamental representation in the present example, the covariant derivative is defined to be

$$(D_\mu \psi)^i = (\partial_\mu \psi^i - i(t^a)^{ij} A^a_\mu \psi^j) \tag{4.10}$$

It can be easily checked that this, indeed, transforms covariantly under the local transformations of the form in eq. (4.3). With this, we can also write the total fermion Lagrangian density more compactly as

$$\mathcal{L}_{TOT} = i\overline{\psi}^i \gamma^\mu (D_\mu \psi)^i \tag{4.11}$$

Observing that the covariant derivative transforms covariantly under the local transformations, the invariance of this Lagrangian density is now obvious. Let us also recall that if^{abc} provides a representation of the generators of $SU(n)$ – the adjoint representation. In fact, it is easy to check, using Jacobi identity, that the exact identification is given by

$$(T^a)^{bc} = -if^{abc} \tag{4.12}$$

From this as well as from the definition of the covariant derivative in eq. (4.9), we note that the change of the gauge fields under the local transformations can also be written as

$$\delta A^a_\mu = (D_\mu \epsilon(x))^a = \partial_\mu \epsilon^a + f^{abc} A^b_\mu \epsilon^c \tag{4.13}$$

where the covariant derivative on the right hand side of eq. (4.13) corresponds to the adjoint representation. This shows that the gauge fields, A^a_μ, belong to the adjoint representation of $SU(n)$. There are, in fact, $n^2 - 1$ components of the gauge fields and, hence, one can also define a matrix valued gauge field as

$$A_\mu = A^a_\mu T^a$$

where, as we have mentioned earlier, T^a's represent the generators in the adjoint representation.

To give dynamics to the gauge fields, we follow exactly what one does in QED. Namely, we define a field strength associated with the gauge fields (potentials) as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \tag{4.14}$$

This, of course, reduces to the form of the Maxwell field strength if we assume the structure constants to vanish (as is true for Abelian theories). We can also scale the gauge fields to bring in a dependence on the coupling constant in the last term in eq. (4.14), which would correspond to the standard form in physical theories. In this form, one would see the essential difference between an Abelian and a non-Abelian field strength tensors in that, because the $SU(n)$ gauge fields carry the internal quantum numbers (generalized charges), they couple to themselves. In contrast, the photon does not carry an electromagnetic charge and hence does not have self-couplings. For simplicity, however, we will work with the form of the field strength in eq. (4.14) which can also be equivalently thought of as corresponding to a unit coupling constant. It is easy to check from the definitions in eqs. (4.13) and (4.14) that for any quantity, $\alpha^a(x)$, belonging to the adjoint representation, we have

$$([D_\mu, D_\nu]\alpha)^a = f^{abc}\, F^b_{\mu\nu}\alpha^c \tag{4.15}$$

The definition of the field strength is such that it transforms covariantly under a gauge transformation of the form in eq. (4.13). This, can be checked from the definition in eq. (4.14) with the help of eq. (4.15) as

$$\begin{aligned} \delta F^a_{\mu\nu} &= (D_\mu \delta A_\nu)^a - (D_\nu \delta A_\mu)^a \\ &= (D_\mu D_\nu \epsilon(x))^a - (D_\nu D_\mu \epsilon(x))^a \\ &= ([D_\mu, D_\nu]\epsilon(x))^a = f^{abc}\, F^b_{\mu\nu}\epsilon^c(x) \end{aligned} \tag{4.16}$$

The form of a Lagrangian density for the gauge fields, which will be at most quadratic in the derivatives and will be gauge invariant, is now easily seen to be (except for the normalization)

$$\mathcal{L}_G = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} \tag{4.17}$$

Gauge invariance is, in fact, obvious, namely,

$$\begin{aligned} \delta\mathcal{L}_G &= -\frac{1}{2} F^{\mu\nu\, a} \delta F^a_{\mu\nu} \\ &= -\frac{1}{2} F^{\mu\nu\, a} f^{abc} F^b_{\mu\nu} \epsilon^c = 0 \end{aligned}$$

where the last step follows because of the antisymmetry of f^{abc} in the a, b indices. Thus, the complete Lagrangian density describing the interactions between fermions and the dynamical non-Abelian gauge fields is given by (see eq. (4.11) and (4.17))

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_G + \mathcal{L}_{TOT} \\ &= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + i\overline{\psi}^i \gamma^\mu (D_\mu \psi)^i \end{aligned} \tag{4.18}$$

While the form of the gauge Lagrangian density in eq. (4.17) is quite similar to that of the Maxwell theory, there are essential differences. For the non-Abelian gauge fields, $\mathcal{L}_G$ represents a fully interacting theory since the gauge fields couple to themselves. This, of course, leads to interesting properties such as asymptotic freedom in such theories. Even quantization of such theories turns out to be nontrivial and that is what we will discuss in the next section.

4.2 BRST Invariance

To understand the quantization of non-Abelian gauge theories, the matter part (fermions) is not very relevant and, therefore,

we would only consider the gauge part of the Lagrangian for the rest of this chapter.

Gauge invariance, as we know from the study of QED, puts a very strong constraint on the structure of the gauge Lagrangian. In particular, the coefficient matrix of the quadratic terms in the Lagrangian density is singular and, therefore, non-invertible. As a result, if we take $\mathcal{L}_G$, as the dynamical theory, then, we cannot define propagators and the entire philosophy of doing perturbative calculations with Feynman diagrams breaks down. To circumvent this difficulty, we normally add to the gauge Lagrangian density a term which breaks gauge invariance and allows us to define the propagator for the gauge fields. Such a term is called a gauge fixing term and any term which makes the coefficient matrix of the quadratic terms nonsingular and maintains various global symmetries of the theory is allowed for this purpose. Thus, for example, the standard covariant gauge fixing consists of adding, to the Lagrangian density, a term of the form

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu A^{\mu a})^2 \tag{4.19}$$

Here ξ represents an arbitrary constant known as the gauge fixing parameter. We note here that it is more convenient, for the later discussions, to write the gauge fixing Lagrangian density by introducing an auxiliary field as

$$\mathcal{L}_{GF} = \partial^\mu F^a A^a_\mu + \frac{\xi}{2} F^a F^a \tag{4.20}$$

It is clear from the form of $\mathcal{L}_{GF}$ in eq. (4.20) that the equation of motion for the auxiliary field takes the form

$$\xi F^a = \partial^\mu A^a_\mu \tag{4.21}$$

and when we eliminate F^a using this equation from eq. (4.20), we recover the original gauge fixing Lagrangian density, eq. (4.19), if we ignore a total divergence term. Among other things $\mathcal{L}_{GF}$ as written in eq. (4.20), allows us to take such gauge choices as the Landau gauge which corresponds to simply taking the limit $\xi = 0$.

By adding a gauge fixing term to the gauge Lagrangian density, we have, of course, modified our theory and, therefore, we must compensate for this change. Conventionally, this is done by adding to the Lagrangian density another term known as the ghost Lagrangian density as follows. Suppose $f^a(x) = 0$ corresponds to the gauge condition described by the gauge fixing Lagrangian density, then, we add to our Lagrangian density another term

$$\mathcal{L}_{ghost} = -\int dy\, \bar{c}^a(x) \left(\frac{\delta f^a(x)}{\delta A^b_\mu(y)}\right) (D_\mu c(y))^b \tag{4.22}$$

where the new fields c^a and $\bar{c}^a$ are known as the ghost and the anti-ghost fields respectively. They are spin zero fields which are anticommuting and are, therefore, unphysical. In general, if the gauge fixing condition involves other fields besides the gauge fields, one can write the ghost action as

$$S_{ghost} = \int dx\, \mathcal{L}_{ghost} = -\int dx dy\, \bar{c}^a(x) \frac{\delta f^{(g)a}(x)}{\delta \epsilon^b(y)} c^b(y) \tag{4.23}$$

where $f^{(g)a}$ represents the gauge transform of the gauge fixing condition and $\epsilon^a(x)$ corresponds to the local parameter of the gauge transformation. In the covariant gauge that we are assuming, for example, we see

from eq. (4.21) that we can write

$$f^a = \partial^\mu A^a_\mu - \xi F^a = 0 \tag{4.24}$$

Consequently, the ghost Lagrangian, for this choice of gauge fixing, follows from eq. (4.22) to be

$$\begin{aligned} \mathcal{L}_{ghost} &= -\int dy\, \bar{c}^a(x) \left(\partial^\mu_x \delta(x-y)\delta^{ab}\right) (D_\mu c(y))^b \\ &= -\bar{c}^a(x)\partial^\mu (D_\mu c(x))^a \\ &= \partial^\mu \bar{c}^a(x)(D_\mu c(x))^a \end{aligned} \tag{4.25}$$

where we have dropped a total derivative term in the last line of the above equation.

With all these modifications, the total Lagrangian density for a gauge theory has the form

$$\mathcal{L}_{TOT} = \mathcal{L}_G + \mathcal{L}_{GF} + \mathcal{L}_{ghost} \tag{4.26}$$

For the covariant gauge that we are using, the total Lagrangian density is given by

$$\mathcal{L}_{TOT} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \partial^\mu F^a A^a_\mu + \frac{\xi}{2} F^a F^a + \partial^\mu \bar{c}^a (D_\mu c)^a \tag{4.27}$$

As we have mentioned earlier, the gauge fixing and the ghost Lagrangian modify the original theory in a compensating manner which allows us to define the Feynman rules of the theory and carry out any perturbative calculation. In a deeper sense, the gauge fixing and the ghost Lagrangians, in the path integral formulation, merely correspond to a

multiplicative factor of unity which does not change the physical content of the theory. However, without going into technical details, we can see that these additional terms in the Lagrangian density have no physical content in the following way.

The total Lagrangian density of eq. (4.26) or more specifically of eq. (4.27), have been gauge fixed and, therefore, do not have the gauge invariance of the original theory. However, the total Lagrangian density, after gauge fixing, develops a global fermionic symmetry which, in some sense, remembers the gauge invariance of the original theory. It is easy to check that the total Lagrangian density is invariant under the global transformations

$$\begin{aligned} \delta A^a_\mu &= \omega (D_\mu c)^a \\ \delta c^a &= -\frac{\omega}{2} f^{abc} c^b c^c \\ \delta \bar{c}^a &= -\omega F^a \\ \delta F^a &= 0 \end{aligned} \tag{4.28}$$

Here ω is the constant anticommuting parameter of the global transformation. Before showing the invariance of the total Lagrangian density under these transformations, it is worth noting that these transformations are nilpotent. Namely, for any field variable, it can be easily checked that

$$\delta_2 \delta_1 \phi^a = 0 \tag{4.29}$$

for $\phi^a = A^a_\mu, F^a, c^a, \bar{c}^a$ independent of the parameters of the transformations. The invariance of the Lagrangian density in eq. (4.27) can now

be easily checked. First, we note that the transformation of A^a_μ is really a gauge transformation with the parameter $\epsilon^a = \omega c^a$ and, therefore, the gauge Lagrangian density is trivially invariant under these transformation, namely,

$$\delta \mathcal{L}_G = 0$$

Consequently, we need to worry only about the gauge fixing and the ghost Lagrangian densities. Furthermore, we note from eq. (4.28) that the auxiliary field, F^a, does not transform at all which gives

$$\begin{aligned} \delta(\mathcal{L}_{GF} + \mathcal{L}_{ghost}) &= \partial^\mu F^a \delta A^a_\mu + \partial^\mu \delta \bar{c}^a (D_\mu c)^a + \partial^\mu \bar{c}^a \delta (D_\mu c)^a \\ &= \omega \partial^\mu F^a (D_\mu c)^a - \omega \partial^\mu F^a (D_\mu c)^a \\ &= 0 \end{aligned} \tag{4.30}$$

In obtaining this result, we have used the fact that $\delta(D_\mu c)^a = 0$ which follows from the nilpotency of the transformations (in particular, for the second variation of the gauge field). We note here, parenthetically, that with the auxiliary fields the transformations are nilpotent off-shell while without them nilpotency would only hold when equations of motion are used.

These transformations which define a residual symmetry of the full theory and, in some sense, replace the original gauge invariance, are known as the BRST transformations and play a fundamental role in the study of gauge theories. There is also a second set of transformations involving the anti-ghost fields of the form

$$\begin{aligned}
\overline{\delta} A^a_\mu &= \overline{\omega}(D_\mu \overline{c})^a \\
\overline{\delta} c^a &= \overline{\omega}(F^a - f^{abc} c^b \overline{c}^c) \\
\overline{\delta} \overline{c}^a &= -\frac{\overline{\omega}}{2} f^{abc} \overline{c}^b \overline{c}^c \\
\overline{\delta} F^a &= \overline{\omega} f^{abc} F^b \overline{c}^c
\end{aligned} \tag{4.31}$$

which leave the full Lagrangian density invariant. These are known as the anti-BRST transformations. However, since these do not lead to any new information beyond what the BRST invariance provides, we will ignore this for the rest of our discussions. In addition to these two anti-commuting symmetries, the total Lagrangian density is also invariant under the bosonic global symmetry transformations

$$\begin{aligned}
\delta c^a &= \theta c^a \\
\delta \overline{c}^a &= -\theta \overline{c}^a
\end{aligned} \tag{4.32}$$

with all other fields remaining unchanged. Here θ represents a constant, bosonic infinitesimal parameter and the generator of the symmetry merely counts the ghost number of the fields. (The fact that these transformations are like scale transformations and not like phase transformations has to do with the particular hermiticity properties that the ghost and the anti-ghost fields must satisfy for consistent quantization of the theory – we will, however, not go into the technical details of this.)

The Hilbert space of the full theory, as is clear by now, contains many more states than the physical states alone. The physical

Hilbert space, therefore, needs to be properly selected out for a discussion of physical questions. Furthermore, the physical space must be selected in such a way that it remains invariant under the time evolution of the system. In the covariant gauge in QED, for example, the physical space is selected to satisfy

$$\partial^\mu A_\mu^{(+)}|\text{phys}\rangle = 0 \tag{4.33}$$

where the superscript, +, stands for the positive frequency part of the field. In QED, this works because $\partial^\mu A_\mu$ satisfies the free Klein-Gordon equation in the covariant gauge and hence the physical space so selected remains invariant under time evolution. The corresponding operator in a non-Abelian theory, on the other hand, does not satisfy a free equation and hence is not suitable for identifying the physical subspace. On the other hand, the generators of the BRST symmetry, Q_{BRST}, and the ghost scaling symmetry, Q_c, are conserved and hence can be used to define a physical Hilbert space which would remain invariant under the time evolution of the system. (Q_{BRST} and Q_c are the charges constructed from the Nöether current for the appropriate transformations whose explicit forms can be obtained from the Nöether currents given later in eq. (4.50).) Thus, we identify the physical space of the theory as satisfying

$$\begin{aligned} Q_{BRST}|\text{phys}\rangle &= 0 \\ Q_c|\text{phys}\rangle &= 0 \end{aligned} \tag{4.34}$$

It can be shown in a straightforward manner that the conditions in eq. (4.34) reduce to the Gupta-Bleuler condition of eq. (4.33) for an Abelian theory.

We are now ready to show that the gauge fixing and the ghost Lagrangian densities lead to no physical consequences. We note from eqs. (4.27) and (4.28) that these extra terms in the Lagrangian density can, in fact, be written as a BRST variation (without the parameter), namely,

$$\begin{aligned} \mathcal{L}_{GF} + \mathcal{L}_{ghost} &= \delta(-\partial^\mu \bar{c}^a A^a_\mu - \frac{\xi}{2}\bar{c}^a F^a) \\ &= [Q_{BRST}, (-\partial^\mu \bar{c}^a A^a_\mu - \frac{\xi}{2}\bar{c}^a F^a)]_+ \end{aligned} \tag{4.35}$$

It now follows from the physical condition in eq. (4.34) that

$$\langle \text{phys}|(\mathcal{L}_{GF} + \mathcal{L}_{ghost})|\text{phys}'\rangle = 0 \tag{4.36}$$

This shows that the terms added to the original Lagrangian density have no contribution to the physical matrix elements of the theory. Second, this also shows that all the physical matrix elements would be independent of the choice of the gauge and the gauge-fixing parameter ξ.

4.3 Ward Identities

The BRST invariance of the full theory leads to many relations between various scattering amplitudes of the theory. These are known as the Ward Identities or the Slavnov-Taylor Identities of the theory and are quite essential in establishing the renormalizability of the gauge theories. These identities are best described within the context of path integrals which we will do next.

Let us consider an effective Lagrangian density which consists of $\mathcal{L}_{TOT}$ as well as source terms as follows.

$$\begin{aligned}\mathcal{L}_{eff} &= \mathcal{L}_{TOT} + J^{\mu a}A^a_\mu + J^a F^a + i(\overline{\eta}^a \mathrm{c}^a - \overline{\mathrm{c}}^a \eta^a) \\ &\quad + K^{\mu a}(D_\mu \mathrm{c})^a + K^a(-\frac{1}{2} f^{abc}\mathrm{c}^b\mathrm{c}^c) \qquad (4.37)\end{aligned}$$

Here, we have not only introduced sources for all the fields in the theory, but, in addition, we have also added sources for the composite variations under a BRST transformation. The need, for this, will become quite clear shortly. Denoting all the fields and sources generically by A and J respectively, we can write the generating functional for the theory as

$$Z[J] = e^{iW[J]} = \int \mathcal{D}A\, e^{i\int d^4x\, \mathcal{L}_{eff}} \qquad (4.38)$$

The vacuum expectation values of operators, in the presence of sources, can now be written as

$$\begin{aligned}\langle A^a_\mu \rangle = A^{(c)a}_\mu &= \frac{\delta W}{\delta J^{\mu a}} \\ \langle F^a \rangle = F^{(c)a} &= \frac{\delta W}{\delta J^a} \\ \langle \mathrm{c}^a \rangle = \mathrm{c}^{(c)a} &= -i\frac{\delta W}{\delta \overline{\eta}^a} \\ \langle \overline{\mathrm{c}}^a \rangle = \overline{\mathrm{c}}^{(c)a} &= -i\frac{\delta W}{\delta \eta^a} \\ \langle (D_\mu \mathrm{c})^a \rangle &= \frac{\delta W}{\delta K^{\mu a}} \\ \langle (-\frac{1}{2} f^{abc}\mathrm{c}^b\mathrm{c}^c) \rangle &= \frac{\delta W}{\delta K^a} \qquad (4.39)\end{aligned}$$

Here, we have assumed a left derivative for the anti-commuting fields. The fields $A^{(c)}$ are known as the classical fields and in what follows, we will ignore the superscript (c) for notational simplicity.

When the external sources are held fixed, the effective Lagrangian density in eq. (4.37) is no longer invariant under the BRST transformations of eq. (4.28). In fact, recalling that $\mathcal{L}_{TOT}$ is BRST invariant and that the BRST transformations are nilpotent, we obtain the change in $\mathcal{L}_{eff}$ (without the parameter – remember that the parameter is anti-commuting) to be

$$\begin{aligned} \delta\mathcal{L}_{eff} &= J^{\mu a}\delta A^a_\mu + J^a\delta F^a + i(\overline{\eta}^a\delta \mathrm{c}^a - \delta\overline{\mathrm{c}}^a\eta^a) \\ &= J^{\mu a}(D_\mu \mathrm{c})^a + i(\frac{1}{2}f^{abc}\overline{\eta}^a\mathrm{c}^b\mathrm{c}^c + F^a\eta^a) \end{aligned} \tag{4.40}$$

On the other hand, the generating functional in eq. (4.38) is defined by integrating over all possible field configurations. Therefore, with a redefinition of the fields under a BRST transformation inside the path integral, the generating functional should be invariant. This immediately leads to

$$\begin{aligned} \delta Z[J] = 0 &= \int \mathcal{D}A\,(i\int d^4x\,\delta\mathcal{L}_{eff})\,e^{i\int d^4x\mathcal{L}_{eff}} \\ &= i\int d^4x\left(J^{\mu a}\langle(D_\mu \mathrm{c})^a\rangle + i\overline{\eta}^a\langle\frac{1}{2}f^{abc}\mathrm{c}^b\mathrm{c}^c\rangle + i\eta^a\langle F^a\rangle\right) \end{aligned}$$

This can also be written using eq. (4.39) as

$$\int d^4x\left(J^{\mu a}(x)\frac{\delta W}{\delta K^{\mu a}(x)} - i\overline{\eta}^a(x)\frac{\delta W}{\delta K^a(x)} + i\eta^a(x)\frac{\delta W}{\delta J^a(x)}\right) = 0 \tag{4.41}$$

This is the Master equation from which one can derive all the identities relating the connected Greens functions of the theory. It is here that the usefulness of the sources for the composite operators becomes obvious.

Most often, however, we are interested in the proper (1PI) vertices of the theory. These can be obtained by passing from the generating functional for the connected Greens functions, $W[J]$, to the generating functional for the proper vertices, $\Gamma[A]$, through a Legendre transformation. Thus, defining a Legendre transformation involving (only) the field variables of the theory (The field variables are really the classical fields and we are dropping the superscript (c) for simplicity.), we have

$$\Gamma[A,K] = W[J] - \int d^4x \, (J^{\mu a} A^a_\mu + J^a F^a + i(\overline{\eta}^a c^a - \overline{c}^a \eta^a)) \qquad (4.42)$$

where K stands generically for the sources for the composite variations. From the definition of the generating functional for the proper vertices in eq. (4.42), it is clear that

$$\begin{aligned}
\frac{\delta\Gamma}{\delta A^a_\mu} &= -J^{\mu a} \\
\frac{\delta\Gamma}{\delta F^a} &= -J^a \\
\frac{\delta\Gamma}{\delta c^a} &= i\overline{\eta}^a \\
\frac{\delta\Gamma}{\delta \overline{c}^a} &= i\eta^a \\
\frac{\delta\Gamma}{\delta K^a_\mu} &= \frac{\delta W}{\delta K^a_\mu} \\
\frac{\delta\Gamma}{\delta K^a} &= \frac{\delta W}{\delta K^a}
\end{aligned} \qquad (4.43)$$

Using the definitions in eq. (4.43), we see that we can rewrite the Master equation of (4.41) in terms of the generating functional of the proper vertices as

$$\int d^4x \left(\frac{\delta\Gamma}{\delta A^a_\mu(x)} \frac{\delta\Gamma}{\delta K^{\mu a}(x)} + \frac{\delta\Gamma}{\delta c^a(x)} \frac{\delta\Gamma}{\delta K^a(x)} - F^a(x) \frac{\delta\Gamma}{\delta \bar{c}^a(x)} \right) = 0 \tag{4.44}$$

This is the Master equation from which we can derive all the relations between various (1PI) proper vertices resulting from the BRST invariance of the theory. This, in turn, is essential in proving the renormalizability of gauge theories. Thus, for example, let us note that we can write eq. (4.44) in the momentum space as

$$\int d^4k \left(\frac{\delta\Gamma}{\delta A^a_\mu(-k)} \frac{\delta\Gamma}{\delta K^{\mu a}(k)} + \frac{\delta\Gamma}{\delta c^a(-k)} \frac{\delta\Gamma}{\delta K^a(k)} - F^a(k) \frac{\delta\Gamma}{\delta \bar{c}^a(k)} \right) = 0 \tag{4.45}$$

Taking derivative of this with respect to $\frac{\delta^2}{\delta F^b(p)\delta c^b(-p)}$ and setting all field variables to zero gives

$$\frac{\delta^2\Gamma}{\delta F^b(p)\delta A^a_\mu(-p)} \frac{\delta^2\Gamma}{\delta c^b(-p)\delta K^{\mu a}(p)} - \frac{\delta^2\Gamma}{\delta c^a(-p)\delta \bar{c}^a(p)} = 0 \tag{4.46}$$

A simple analysis, then, shows that eq. (4.46) relates the mixed two point function involving F-A_μ with the two point function for the ghost fields and, consequently, the counter terms should satisfy such a relation. The BRST invariance, in this way, is very fundamental in the study of gauge theories as far as renormalizability and gauge independence are concerned.

4.4 Unitarity

The BRST invariance of a gauge theory is quite important from yet another consideration. It leads to a formal proof of unitarity of the theory when restricted to the physical subspace of the Hilbert space. We will only outline the proof of this in this section.

From the Lagrangian density in eq. (4.27), we can obtain the canonical momenta conjugate to the various field variables of the theory which are given by

$$\begin{aligned}
\pi^{ia} &= \frac{\partial \mathcal{L}}{\partial \dot{A}_i^a} = -F^{0ia} \\
\pi^a &= \frac{\partial \mathcal{L}}{\partial \dot{F}^a} = A_0^a \\
\pi_c^a &= \frac{\partial \mathcal{L}}{\partial \dot{c}^a} = -\dot{\bar{c}}^a \\
\pi_{\bar{c}}^a &= \frac{\partial \mathcal{L}}{\partial \dot{\bar{c}}^a} = (D_0 c)^a
\end{aligned} \tag{4.47}$$

Here we have used left derivatives for the anti-commuting ghost fields. In particular, we see that A_0^a plays the role of the momentum conjugate to F^a. The canonical (anti) commutation relations can now be written as ($\hbar = 1$)

$$\begin{aligned}
\left[A_i^a(\vec{x},t), \pi^{jb}(\vec{y},t)\right] &= i\delta^{ab}\delta_i^j \delta^3(x-y) \\
\left[F^a(\vec{x},t), \pi^b(\vec{y},t)\right] &= i\delta^{ab}\delta^3(x-y) \\
\left[c^a(\vec{x},t), \pi_c^b(\vec{y},t)\right]_+ &= i\delta^{ab}\delta^3(x-y) \\
\left[\bar{c}^a(\vec{x},t), \pi_{\bar{c}}^b(\vec{y},t)\right]_+ &= i\delta^{ab}\delta^3(x-y)
\end{aligned} \tag{4.48}$$

The hermiticity conditions for the ghost fields which arise out of various consistency conditions are

$$\begin{aligned} (\mathrm{c}^a)^\dagger &= \mathrm{c}^a \\ (\bar{\mathrm{c}}^a)^\dagger &= -\bar{\mathrm{c}}^a \end{aligned} \tag{4.49}$$

The Nöether current densities associated with the BRST symmetry and the ghost scaling symmetry (see eqs. (4.28) and (4.32)) can be obtained in a straightforward manner and have the forms

$$\begin{aligned} J^{\mu a}_{BRST} &= F^{\mu\nu a}(D_\nu \mathrm{c})^a + F^a(D^\mu \mathrm{c})^a - \frac{1}{2} f^{abc} \partial^\mu \bar{\mathrm{c}}^a \mathrm{c}^b \mathrm{c}^c \\ J^\mu_c &= \partial^\mu \bar{\mathrm{c}}^a \mathrm{c}^a - \bar{\mathrm{c}}^a (D^\mu \mathrm{c})^a \end{aligned} \tag{4.50}$$

The corresponding conserved charges can also be obtained from these current densities and we can calculate the algebra of charges using the (anti) commutation relations in eq. (4.48). We note, in particular, that

$$\begin{aligned} [\mathrm{c}^a(x), Q_c] &= -i\mathrm{c}^a(x) \\ [\bar{\mathrm{c}}^a(x), Q_c] &= i\bar{\mathrm{c}}^a(x) \end{aligned} \tag{4.51}$$

Therefore, we can think of iQ_c as the ghost number operator (with the ghost number for $\bar{\mathrm{c}}^a$ being negative).

The algebra of the charges also follows in a straightforward manner.

$$
\begin{aligned}
[Q_{BRST}, Q_{BRST}]_+ &= 0 \\
[Q_c, Q_c] &= 0 \\
[Q_{BRST}, Q_c] &= -iQ_{BRST}
\end{aligned}
\tag{4.52}
$$

The first equation simply reiterates in an operator language the fact that the BRST transformations are nilpotent. The second equation implies that the ghost scaling symmetry is Abelian. The third merely is a statement of the fact that the BRST charge carries a unit ghost number. An immediate consequence of the third equation in (4.52) is that

$$
\begin{aligned}
Q_{BRST}e^{\pi Q_c} &= e^{\pi(Q_c - i)}Q_{BRST} = -e^{\pi Q_c}Q_{BRST} \\
\text{or, } \left[Q_{BRST}, e^{\pi Q_c}\right]_+ &= 0
\end{aligned}
\tag{4.53}
$$

This relation, as we will see later, is quite useful in dealing with gauge theories at finite temperature.

As is clear from our earlier discussions, the total vector space, V, of the complete gauge theory contains various unphysical states as well as states with negative norm. Consequently, the metric of this space and, therefore, the inner product becomes indefinite. A probabilistic description of the quantum theory, therefore, is lost unless we can restrict to a suitable subspace with a positive definite inner product. As we have discussed earlier, we can select out a subspace, V_{phys}, by requiring that states in this space are annihilated by Q_{BRST}. Namely, for every $|\Psi\rangle \in V_{\text{phys}}$, we have

$$
Q_{BRST}|\Psi\rangle = 0 \tag{4.54}
$$

We note again that such an identification is invariant under the time evolution of the system since Q_{BRST} commutes with the Hamiltonian because the BRST transformations are a symmetry of the full theory, namely,

$$[Q_{BRST}, H] = 0$$

Furthermore, there are two possible kinds of states which will satisfy the condition in eq. (4.54). First, if a state Ψ cannot be written as

$$|\Psi\rangle \neq Q_{BRST}|\tilde{\Psi}\rangle \tag{4.55}$$

and still satisfies eq. (4.54), then, it is truly a BRST singlet state. The field operator which creates such a state must necessarliy commute with Q_{BRST} and, therefore, must represent a truly gauge invariant field variable, namely, (We assume that the vacuum state is a BRST singlet state.)

$$\begin{aligned} Q_{BRST}|\Psi\rangle &= Q_{BRST}\Psi|0\rangle \\ &= [Q_{BRST}, \Psi]\,|0\rangle = 0 \end{aligned} \tag{4.56}$$

where we have assumed that the operator, Ψ, creates the state $|\Psi\rangle$ from vacuum, namely,

$$\Psi\,|0\rangle = |\Psi\rangle \tag{4.57}$$

Equation (4.56) also implies that

$$[Q_{BRST}, \Psi] = 0 \tag{4.58}$$

Such states would, therefore, correspond to truly physical states of the theory and since gauge invariant degrees of freedom have physical (non-negative) commutation relations, such states would have positive norm.

The second class of states which would satisfy eq. (4.54) can be of the form

$$|\Psi\rangle = Q_{BRST}|\tilde{\Psi}\rangle \tag{4.59}$$

We see that the condition (4.54) is trivially satisfied in this case because of the nilpotency of Q_{BRST}. The nilpotency of Q_{BRST} also implies that all such states would have zero norm because (Q_{BRST} is Hermitian with our choice of hermiticity conditions for the ghost fields.)

$$\langle\Psi|\Psi'\rangle = \langle\tilde{\Psi}|Q_{BRST}\,Q_{BRST}|\tilde{\Psi}'\rangle = 0$$

Such a state would also be orthogonal to a physical state of the form in eqs. (4.55) and (4.54) because if $|\Psi\rangle$ represents a truly physical state while $|\Psi'\rangle$ a state of the kind in eq. (4.59), then

$$\langle\Psi'|\Psi\rangle = \langle\tilde{\Psi}'|Q_{BRST}|\Psi\rangle = 0$$

which follows because of eq. (4.54). If we denote all the states of $V_{\rm phys}$ of the form as in eq. (4.59) by V_0, then, this would contain all the zero norm states and would be orthogonal to $V_{\rm phys}$ itself. The true physical states, which satisfy eqs. (4.54) and (4.55) can, therefore, be identified as belonging to the quotient space $\overline{V}_{\rm phys} = \frac{V_{\rm phys}}{V_0}$ and will have positive definite norm.

Every state in $V_{\rm phys}$ can, of course, be projected out according to the number of unphysical particles. Thus, defining $P^{(n)}$ as the projection operator onto the n-unphysical particle sector, we have

$$\begin{aligned} \sum_{n=0}^{\infty} P^{(n)} &= 1 \\ P^{(n)}P^{(m)} &= \delta_{n,m}P^{(n)} \end{aligned} \tag{4.60}$$

It is clear that, by definition, $P^{(0)}$ projects onto the truly physical states of the theory and from the first equation of (4.60), we note that we can write

$$P^{(0)} = 1 - P' \tag{4.61}$$

where

$$P' = \sum_{n=1}^{\infty} P^{(n)} \tag{4.62}$$

One can, of course, construct the actual forms of all the projection operators which is not very interesting. What is interesting, however, is the fact that $P^{(0)}$ projects onto the true physical states and, therefore, must be gauge invariant and by eq. (4.58), it will commute with Q_{BRST}. It follows, then, from eq. (4.62) that P' must also commute with Q_{BRST}. Namely, we have

$$\begin{aligned} \left[Q_{BRST}, P^{(0)}\right] &= 0 \\ \left[Q_{BRST}, P'\right] &= 0 \end{aligned} \tag{4.63}$$

The difference, however, is that P' must necessarily involve unphysical fields since it projects onto unphysical states and, therefore, cannot be truly gauge invariant. Consequently, it must have the form (for some operator, R, whose explicit form is not important for our discussions)

$$P' = [Q_{BRST}, R]_+ \tag{4.64}$$

which would imply eq. (4.63) because of the nilpotency of Q_{BRST}. In fact, one can show more than this, namely, one can show that for $P^{(n)}$, $n \geq 1$, we can write

$$P^{(n)} = \left[Q_{BRST}, R^{(n)}\right]_+ \qquad \text{for } n \geq 1 \tag{4.65}$$

It is now easy to show that $P^{(n)}$, for $n \geq 1$, and, therefore, P', projects onto the zero norm space, V_0. For example, let $|\Psi\rangle, |\Psi'\rangle \in V_{\rm phys}$, then,

$$\langle\Psi'|P^{(n)}|\Psi\rangle = \langle\Psi'|\left[Q_{BRST}, R^{(n)}\right]_+|\Psi\rangle = 0 \tag{4.66}$$

which follows from the physical state condition in eq. (4.54). Any vector, $|\Psi\rangle \in V_{\rm phys}$ can now be expanded as

$$|\Psi\rangle = P^{(0)}|\Psi\rangle + P'|\Psi\rangle \tag{4.67}$$

and the norm of any such state is, then, obtained to be

$$\langle\Psi|\Psi\rangle = \langle\Psi|P^{(0)}|\Psi\rangle \geq 0 \tag{4.68}$$

This, therefore, shows that $V_{\rm phys}$, defined by eq. (4.54), has a positive semi-definite norm as we should have for a physical vector space and the value of the norm depends on the truly physical component of the state. This also makes it clear that the norm of a state $|\overline{\Psi}\rangle \in \overline{V}_{\rm phys} = \frac{V_{\rm phys}}{V_0}$ would have a positive definite norm. This would correspond to the true physical subspace of the total vector space. We also note here that for $|\Psi\rangle, |\Phi\rangle \in V_{\rm phys}$ and any two operators, A and B, in $V_{\rm phys}$ (*i.e.*, any two operators which do not take us out of $V_{\rm phys}$)

$$\begin{aligned}\langle\Psi|P^{(0)}|\Phi\rangle &= \langle\Psi|(1-P')|\Phi\rangle = \langle\Psi|\Phi\rangle \\ \langle\Psi|AP^{(0)}B|\Phi\rangle &= \langle\Psi|A(1-P')B|\Phi\rangle = \langle\Psi|AB|\Phi\rangle\end{aligned} \tag{4.69}$$

which follow from the condition in eq. (4.54).

The formal proof of unitarity now follows. First, we note that, with the hermiticity assignments of eq. (4.49) for the ghost fields,

the complete gauge Lagrangian of eq. (4.27) is hermitian. Consequently, the S-matrix of the theory is formally unitary, namely,

$$S^\dagger S = S S^\dagger = 1 \tag{4.70}$$

Furthermore, the S-matrix will be BRST invariant since the full theory is, so that

$$[Q_{BRST}, S] = 0 \tag{4.71}$$

We note next that given any state $|\Psi\rangle \in V_{\rm phys}$, we can define a unique state $|\overline{\Psi}\rangle \in \overline{V}_{\rm phys}$ as follows

$$|\overline{\Psi}\rangle = P^{(0)}|\Psi\rangle \tag{4.72}$$

In the space $\overline{V}_{\rm phys}$, of course, $P^{(0)}$ acts as the identity operator, namely,

$$P^{(0)}|\overline{\Psi}\rangle = (P^{(0)})^2|\Psi\rangle = P^{(0)}|\Psi\rangle = |\overline{\Psi}\rangle \tag{4.73}$$

The two states, $|\Psi\rangle$ and $|\overline{\Psi}\rangle$, differ only by a zero norm state which is orthogonal to every state in $V_{\rm phys}$ so that the inner product of any two such states is the same.

$$\langle\overline{\Psi}|\overline{\Phi}\rangle = \langle\Psi|(P^{(0)})^2|\Phi\rangle = \langle\Psi|P^{(0)}|\Phi\rangle = \langle\Psi|\Phi\rangle$$

where the last line follows from eq. (4.69). Given this, it is clear that we can define the S-matrix which acts on the physical space $\overline{V}_{\rm phys}$ as satisfying

$$P^{(0)} S = S_{\rm phys} P^{(0)} \tag{4.74}$$

such that for $|\Psi\rangle \in V_{\rm phys}$, $|\overline{\Psi}\rangle \in \overline{V}_{\rm phys}$ and related through eq. (4.72),

$$S_{\rm phys}|\overline{\Psi}\rangle = S_{\rm phys}P^{(0)}|\Psi\rangle = P^0 S|\Psi\rangle = \overline{S|\Psi\rangle} \tag{4.75}$$

It is clear that since S is BRST invariant (eq. (4.71)), it will leave the space $V_{\rm phys}$ invariant. Furthermore, from eqs. (4.69)-(4.75), we obtain

$$\begin{aligned}\langle\overline{\Psi}|S^\dagger_{\rm phys}S_{\rm phys}|\overline{\Phi}\rangle &= \langle\Psi|P^{(0)}S^\dagger_{\rm phys}S_{\rm phys}P^{(0)}|\Phi\rangle \\ &= \langle\Psi|S^\dagger P^{(0)}S|\Phi\rangle \\ &= \langle\Psi|S^\dagger S|\Phi\rangle = \langle\Psi|\Phi\rangle = \langle\overline{\Psi}|\overline{\Phi}\rangle \end{aligned} \tag{4.76}$$

where we have used the formal unitary nature of S from eq. (4.70). It follows now that

$$S^\dagger_{\rm phys}S_{\rm phys} = 1 \tag{4.77}$$

In other words, the physical state condition which naturally follows from the BRST invariance of the theory, also automatically leads to a formal proof of unitarity of the S-matrix in the subspace of the truly physical states of the theory.

4.5 Fermionic Boundary Conditions

As we have noted earlier in chapter 1, the partition function involving fermionic variables is defined with an anti-periodic boundary condition for the fermionic variables. We will outline in this section how this arises with a simple example of the fermionic quantum mechanical oscillator. Furthermore, we will describe the system in the

coherent state representation which generalizes naturally to the path integral description for field theories.

Let us consider a fermionic oscillator described by the annihilation and creation operators, a and $a^\dagger$ respectively, satisfying the anti-commutation relations

$$\begin{aligned} [a, a]_+ &= 0 = \left[a^\dagger, a^\dagger\right]_+ \\ \left[a, a^\dagger\right]_+ &= 1 \end{aligned} \tag{4.78}$$

The Hilbert space, in this case, is two dimensional with the basis states $|0\rangle$ and $|1\rangle = a^\dagger|0\rangle$ satisfying the orthonormality and the completeness relations given by

$$\begin{aligned} \langle n|m\rangle &= \delta_{nm} \qquad n, m = 0, 1 \\ |0\rangle\langle 0| + |1\rangle\langle 1| &= 1 \end{aligned} \tag{4.79}$$

Let us next consider a Grassmann odd parameter ψ and define a state of the theory as

$$|\psi\rangle = e^{-\psi a^\dagger}|0\rangle = |0\rangle - \psi|1\rangle \tag{4.80}$$

Here, we have used the fact that ψ is Grassmann odd and, therefore, higher powers of ψ vanish. It also follows from eq. (4.80) that

$$\langle\psi| = \langle 0|e^{-a\psi^*} = \langle 0| + \psi^*\langle 1| \tag{4.81}$$

In deriving eq. (4.81), we have used the fact that ψ (or ψ^*) anti-commutes with the fermionic operators of the theory and that

$$(\psi a^\dagger)^\dagger = a\psi^*$$

which is essential to show that for any two such states

$$\langle\phi|\psi\rangle = (\langle\psi|\phi\rangle)^*$$

The states defined in eqs. (4.80) are known as fermionic coherent states and are eigenstates of the fermionic annihilation operator

$$a|\psi\rangle = a(|0\rangle - \psi|1\rangle) = \psi|0\rangle = \psi|\psi\rangle$$

Coherent states are an alternate method of describing the fermion oscillator Hilbert space and as we have mentioned earlier lead naturally to a path integral description of field theories. The inner product of two fermionic coherent states can be easily worked out. For ψ and ϕ Grassmann odd, we have from eqs. (4.80) and (4.81)

$$\begin{aligned}\langle\psi|\phi\rangle &= (\langle 0| + \psi^*\langle 1|)(|0\rangle - \phi|1\rangle) \\ &= \langle 0|0\rangle + \psi^*\phi\langle 1|1\rangle = 1 + \psi^*\phi = e^{\psi^*\phi} \qquad (4.82)\end{aligned}$$

Once again, in obtaining this, we have used the fact that the Grassmann odd parameters anti-commute with fermionic operators.

The completeness relation of the coherent states is obtained, again noting from eqs. (4.80) and (4.81), that

$$|\psi\rangle\langle\psi| = |0\rangle\langle 0| - \psi|1\rangle\langle 0| + \psi^*|0\rangle\langle 1| - \psi^*\psi|1\rangle\langle 1|$$

Therefore, it now follows that

$$\begin{aligned}
&\int d\psi^* d\psi\, e^{-\psi^*\psi}|\psi\rangle\langle\psi| \\
&= \int d\psi^* d\psi\, (1-\psi^*\psi)(|0\rangle\langle 0| - \psi|1\rangle\langle 0| + \psi^*|0\rangle\langle 1| - \psi^*\psi|1\rangle\langle 1|) \\
&= \int d\psi^* d\psi\, (-\psi^*\psi)(|0\rangle\langle 0| + |1\rangle\langle 1|) = 1
\end{aligned} \tag{4.83}$$

In deriving this relation, we have used the nilpotency of the Grassmann odd parameters as well as the Berezin integration rules for Grassmann odd parameters given by

$$\begin{aligned}
\int d\psi\, 1 &= 0 \\
\int d\psi\, \psi &= 1
\end{aligned} \tag{4.84}$$

It can be easily checked now (for example, using eq. (4.82)), that

$$\begin{aligned}
\int d\psi^* d\psi\, e^{-\psi^*\psi}\, |\psi\rangle\langle\psi|\psi'\rangle &= \int d\psi^* d\psi\, e^{-\psi^*\psi}\, e^{\psi^*\psi'}|\psi\rangle \\
&= \int d\psi^* d\psi\, e^{-\psi^*(\psi-\psi')}\, |\psi\rangle \\
&= \int d\psi\, \delta(\psi-\psi')\, |\psi\rangle = |\psi'\rangle
\end{aligned}$$

It is clear that in the coherent state description, the Gaussian factor $e^{-\psi^*\psi}$ acts like a weight factor for various quantities.

With all these, we can now calculate the trace of a Bosonic operator, A, using eq. (4.83) as follows.

$$\begin{aligned} \mathrm{Tr}A &= \mathrm{Tr}1A = \mathrm{Tr}\int d\psi^* d\psi\, e^{-\psi^*\psi}\, |\psi\rangle\langle\psi|A \\ &= \int d\psi^* d\psi\, e^{-\psi^*\psi}\, \langle\psi|A|-\psi\rangle \end{aligned} \tag{4.85}$$

The last line in eq. (4.85) is again due to the anti-commuting nature of the Grassmann odd parameters. That eq. (4.85) is, indeed, correct can be verified explicitly using eqs. (4.80) and (4.81) as follows.

$$\begin{aligned} &\int d\psi^* d\psi\, e^{-\psi^*\psi}\, \langle\psi|A|-\psi\rangle = \\ &\int d\psi^* d\psi\, (1-\psi^*\psi)(\langle 0|A|0\rangle + \psi^*\langle 1|A|0\rangle + \psi\langle 0|A|1\rangle - \psi^*\psi\langle 1|A|1\rangle) \\ &= \int d\psi^* d\psi\, (-\psi^*\psi)(\langle 0|A|0\rangle + \langle 1|A|1\rangle) \\ &= \langle 0|A|0\rangle + \langle 1|A|1\rangle = \mathrm{Tr}A \end{aligned} \tag{4.86}$$

The relation in eq. (4.85) is quite significant in that it says that the trace of a bosonic operator in a fermionic theory is defined only over anti-periodic states. This is the reason for the anti-periodicity in the definition of the partition function for a fermionic theory.

We note here, for later use, that the number operator for the fermions is idempotent as can be checked from the anti-commutation relations in eq. (4.78), namely,

$$N = a^\dagger a = N^2 \tag{4.87}$$

Consequently, it follows that

$$e^{-i\pi N} = 1 - 2N \tag{4.88}$$

This, then, leads from the definition of the coherent states in eq. (4.80) to

$$e^{-i\pi N}|\psi\rangle = (1-2N)(|0\rangle - \psi|1\rangle) = |0\rangle + \psi|1\rangle = |-\psi\rangle \tag{4.89}$$

As a result, for a bosonic operator of the form $e^{-i\pi N}A$, the trace in a fermionic space would have the form

$$\int d\psi^* d\psi \; e^{-\psi^*\psi}\langle\psi|e^{-i\pi N}A|-\psi\rangle = \int d\psi^* d\psi \; e^{-\psi^*\psi}\langle-\psi|A|-\psi\rangle \tag{4.90}$$

In other words, the operator $e^{-i\pi N}$ corresponds to a chemical potential for the fermions which is

$$\mu = i\pi$$

whose presence in a trace changes the boundary conditions for the fermions so that they end up obeying periodic boundary conditions much like bosons.

4.6 Partition Function

With all these background, we are now ready to define the partition function for a gauge theory. From a practical point of view, if we can give the partition function a path integral representation, it would be useful from the point of view of calculations. However, if we try to do that naively, we run into an immediate problem.

As we have seen, the complete theory for a gauge system involves unphysical particles (see eq. (4.27)) and that the BRST

invariance plays a very crucial role in the study of gauge theories from various points of view. However, we note that the ghost fields, c and $\bar{c}$, which effectively behave like the parameters of the BRST transformation, eq. (4.28), are anti-commuting Grassmann fields. They have the statistics of fermions even though they are spin zero fields. Consequently, we expect that at finite temperature they must satisfy anti-periodicity properties like any other fermion. However, the gauge fields, being bosonic, would naturally obey periodic boundary conditions at finite temperature. Thus, we see that there is a problem as far as the BRST transformations of eq. (4.28) are concerned. With these periodicity properties, we note that the left hand side and the right hand side of eq. (4.28) are not consistent. Therefore, if we accept the periodicity properties as following from the statistics of the various fields, we may have to give up BRST invariance. But this symmetry is too valuable to give up. Consequently, we see that if we want to preserve the BRST invariance of the theory, we have to assume a periodic boundary condition for the unphysical ghost fields even though they satisfy anti-commutation relations like fermions. We will see now how this arises from the physical state condition of the theory.

In dealing with a gauge theory, we see that it contains unphysical particles which cannot thermalize in a heat bath. Therefore, it is not meaningful to define a partition function as a trace over the complete vector space of the theory. Rather, the partition function leads to physical observables and, therefore, should be defined as a trace over the physical states of the theory. Thus, it is proper to define the partition function for a gauge theory as

$$Z(\beta) = \overline{\mathrm{Tr}}\, e^{-\beta H} \tag{4.91}$$

Furthermore, recalling that $P^{(0)}$ projects onto the physical subspace of states and that such states have a zero ghost number, we can also rewrite this by relaxing the sum over the physical states as

$$Z(\beta) = \mathrm{Tr}\, P^{(0)}\, e^{-\beta H} = \mathrm{Tr}\, P^{(0)}\, e^{\pi Q_c}\, e^{-\beta H} \tag{4.92}$$

Here, in the second step, we have used the fact that the physical states have zero ghost number. This is a very simple fact, but from the point of view of calculations, it is quite useful. To rewrite the partition function in a more familiar form, we use eqs. (4.61), (4.62) and (4.64) as well as (4.53) to obtain

$$\begin{aligned} Z(\beta) &= \mathrm{Tr}\, P^{(0)}\, e^{\pi Q_c}\, e^{-\beta H} \\ &= \mathrm{Tr}\, (1 - P')\, e^{\pi Q_c}\, e^{-\beta H} \\ &= \mathrm{Tr}\, \left(1 - [Q_{BRST}, R]_+\right)\, e^{\pi Q_c}\, e^{-\beta H} \\ &= \mathrm{Tr}\, \left(e^{\pi Q_c}\, e^{-\beta H} - R\left[Q_{BRST}, e^{\pi Q_c}\right]_+\, e^{-\beta H}\right) \\ &= \mathrm{Tr}\, e^{\pi Q_c}\, e^{-\beta H} \end{aligned} \tag{4.93}$$

In writing the terms in specific forms, we have used the cyclicity of the trace in addition to the fact that the Hamiltonian commutes with Q_{BRST} and Q_c which generate symmetries of the theory as well as eq. (4.53). Finally, we note from eq. (4.51) that iQ_c really corresponds to N_c – the ghost number operator. Hence it follows from eq. (4.90) that the effect of the first exponential in eq. (4.93) is to change the periodicity of the ghost fields and, therefore, we can write

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H} \tag{4.94}$$

where the ghost fields are supposed to satisfy periodic boundary conditions. This is, of course, consistent with our expectation from the BRST invariance point of view. Once, we recognize that we can write the partition function over the unrestricted space as in eq. (4.94), a standard path integral representation follows where the ghosts satisfy periodic boundary condition. We note here, parenthetically, that for any bosonic operator, A, following the same steps as in eq. (4.93), we can write

$$\begin{aligned} \langle A \rangle_\beta &= Z^{-1}(\beta)\ \mathrm{Tr}\ P^{(0)}\ e^{\pi Q_c}\ e^{-\beta H}\ A \\ &= Z^{-1}(\beta)\ \mathrm{Tr}\left(e^{\pi Q_c}\ e^{-\beta H}\ (A - [A, Q_{BRST}]\, R)\right) \end{aligned}$$

This reduces, for physical, BRST invariant observables, to the familiar form (namely, for operators, A, satisfying $[Q_{BRST}, A] = 0$)

$$\langle A \rangle_\beta = Z^{-1}(\beta)\ \mathrm{Tr}\ e^{\pi Q_c}\ e^{-\beta H}\ A$$

which can be written in the more familiar form with periodic ghost fields. However, we also note that for BRST non-invariant operators, the extra term on the right hand side cannot be neglected.

To conclude this section, we note that the presence of ghosts satisfying periodic boundary conditions in the path integral representation is absolutely crucial for a meaningful, gauge independent definition of the partition function. This is true even in an Abelian gauge theory where the ghosts are noninteracting. To see this, let us note that, if we simplify the full theory of eq. (4.27) by eliminating the auxiliary fields, in the Abelian case, it has the form

$$\begin{aligned}\mathcal{L}_{TOT} &= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \partial^\mu \bar{c}\partial_\mu c \\ &= \frac{1}{2} A_\mu \left(\eta^{\mu\nu}\Box - \frac{\xi-1}{\xi}\partial^\mu\partial^\nu\right) A_\nu + \partial^\mu \bar{c}\partial_\mu c \qquad (4.95)\end{aligned}$$

Therefore, if we write the partition function as a path integral assuming that the ghost fields satisfy periodic boundary conditions, we obtain

$$\begin{aligned}Z(\beta) &= N \int \mathcal{D}A_\mu \, \mathcal{D}\bar{c} \, \mathcal{D}c \, e^{-\int_0^\beta dt \int d^3x \, \mathcal{L}_{TOT}} \\ &= N \left(\det(\eta^{\mu\nu}\,\Box - \frac{\xi-1}{\xi}\partial^\mu\partial^\nu)\right)^{-1/2} \det(-\Box) \qquad (4.96)\end{aligned}$$

Here N is a normalization constant and the first determinant in eq. (4.96) arises from the integration of the gauge fields and the determinant is over the finite dimensional Minkowski space (μ, ν) as well as the infinite dimensional coordinate space. The second determinant is from the integration of the ghost variables and the positive power is because of the anti-commuting nature of these fields. Because of our choice of periodic boundary conditions for the ghost fields, both these determinants are defined over the space of periodic functions.

The first determinant can be evaluated over the finite dimensional Minkowski space and a little bit of work shows that it can be written as (it has to be a Lorentz scalar) $\det\left(\frac{\Box^4}{\xi}\right)$. The temperature independent constant, ξ, can be absorbed into the normalization constant so that we can write the partition function as

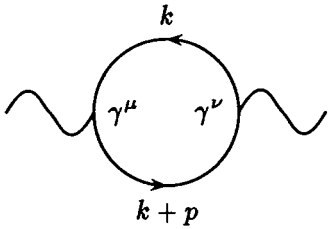
$$Z(\beta) = N \, (\det(\Box^4))^{-1/2} \, \det(-\Box) = N \, \left((\det(-\Box))^{-1/2}\right)^2 \qquad (4.97)$$

We note that this partition function is exactly identical to that of two scalar fields implying that the gauge theory only has two physical trans-

verse degrees of freedom as one can explicitly verify in the unitary gauge. The important thing is that this follows only because the ghost determinant cancelled out exactly the two unphysical degrees of freedom (the longitudinal and the timelike) and this was possible because of the periodic boundary conditions assigned to the ghost fields. Furthermore, the ghost contribution to the partition function is nontrivial and quite crucial even though the ghost fields are noninteracting in this theory. Finally, we note that even though we worked out the partition function in the covariant gauge, the same result is obtained in any other gauge.

4.7 Ward Identities at $T \neq 0$

As is clear from the discussions in this chapter, in defining the partition function of a gauge theory, we have taken care to preserve the BRST symmetry of the zero temperature theory which has many physical consequences. We have also seen that it is the BRST invariance of the complete, gauge fixed theory which leads to the Ward identities of the theory relating various 1PI amplitudes. Therefore, it is clear that the Ward identities will continue to hold even at finite temperature. Let us illustrate this with a simple example. Let us analyze the one loop correction to the polarization tensor of a gauge field, due to the fermion loop, at finite temperature in $1+1$ dimensions. At zero temperature, we know that this is completely transverse (when regularized in a gauge invariant manner using, say, dimensional regularization and this follows from the Ward identities) and what we would like to see is if finite temperature corrections lead to any longitudinal components of the polarization tensor which would violate Ward identities.



The question of Ward identities is much better studied in a real time formalism and we note from eq. (2.45) that, at finite temperature, the Feynman propagator for the physical fermion fields is given by (We suppress the subscript ++ for simplicity and note that, at one loop, the other components of the propagator will not contribute to the physical polarization tensor.)

$$\begin{aligned} iS(p) &= iS^{(0)}(p) + iS^{(\beta)}(p) \\ &= i(\not{p} + m)\left(\frac{1}{p^2 - m^2 + i\epsilon} + 2i\pi n_F(p)\delta(p^2 - m^2)\right) \end{aligned} \quad (4.98)$$

Here $n_F(p)$ represents the fermionic distribution function defined in the covariant notation of chapter 1 (see eq. (1.77)) to be

$$n_F(p) = \frac{1}{e^{\beta|u\cdot p|} + 1}$$

where u^μ, as in chapter 1, represents the velocity of the heat bath.

The temperature dependent contribution of the fermion loop to the polarization tensor can now be easily calculated and has the form

$$\begin{aligned} i\Pi^{(\beta)}_{\mu\nu}(p) &= -e^2 \int \frac{d^2k}{(2\pi)^2} \mathrm{Tr}\left[\gamma_\mu S^{(0)}(k+p)\gamma_\nu S^{(\beta)}(k)\right. \\ &\quad \left. + \gamma_\mu S^{(\beta)}(k+p)\gamma_\nu S^{(0)}(k) + \gamma_\mu S^{(\beta)}(k+p)\gamma_\nu S^{(\beta)}(k)\right] \end{aligned}$$

$$
\begin{aligned}
= & -\frac{ie^2}{\pi}\int d^2k \\
& \times \left[(k+p)_\mu k_\nu + (k+p)_\nu k_\mu - \eta_{\mu\nu}(k\cdot(k+p) - m^2)\right] \\
& \times \left[\frac{n_F(k)\delta(k^2-m^2)}{(k+p)^2 - m^2 + i\epsilon} + \frac{n_F(k+p)\delta((k+p)^2-m^2)}{k^2 - m^2 + i\epsilon}\right. \\
& \left. + 2i\pi n_F(k) n_F(k+p)\delta(k^2-m^2)\delta((k+p)^2-m^2)\right]
\end{aligned}
\tag{4.99}
$$

Here we have used the fact that the temperature dependent terms are finite and, as a result, we have performed the Dirac trace in $1+1$ dimensions.

The simplicity of $1+1$ dimensions allows us to factor the tensor structure out of the integral. In fact, let us define, following the discussion of chapter 1 (section 1.8),

$$
\begin{aligned}
\Omega &= k\cdot u \qquad\qquad \omega = p\cdot u \\
k^\mu &= \Omega u^\mu - \epsilon^{\mu\nu}u_\nu k' \\
p^\mu &= \omega u^\mu - \epsilon^{\mu\nu}u_\nu p'
\end{aligned}
\tag{4.100}
$$

Here, $\epsilon^{\mu\nu}$ defines the Levi-Civita tensor of $1+1$ dimensions and in our notation, $\epsilon^{01} = 1$. We note, for later use, that eq. (4.100) also gives

$$
\epsilon^{\mu\nu}p_\mu u_\nu = p' \tag{4.101}
$$

Substituting these definitions into eq. (4.99), and using the delta function constraints as well as the symmetry properties of the integral, we

find, in a straightforward manner, that the integral can be written as (The Jacobian of the transformation can be easily checked to be unity.)

$$i\Pi^{(\beta)}_{\mu\nu}(p) = -e^2 \bar{u}_\mu \bar{u}_\nu (2I + I') \tag{4.102}$$

where we have defined the velocity transverse to the momentum, p, as (This can be checked easily with eqs. (4.100) and (4.101).)

$$\bar{u}_\mu = u_\mu - \frac{\omega}{p'} \epsilon_{\mu\nu} u^\nu \tag{4.103}$$

and I and I' stand for the integrals

$$\begin{aligned} I &= \frac{ip'}{\omega} \int d\Omega dk' \frac{k'(\omega+\Omega) + \Omega(k'+p')}{\omega(\omega+2\Omega) - p'(p'+2k') + i\epsilon} \frac{1}{e^{\beta|\Omega|}+1} \\ &\quad \times \delta(\Omega^2 - k'^2 - m^2) \\ I' &= -\frac{2p'}{\omega} \int d\Omega dk' \frac{k'(\omega+\Omega) + \Omega(k'+p')}{(e^{\beta|\Omega|}+1)(e^{\beta|\omega+\Omega|}+1)} \delta(\Omega^2 - k'^2 - m^2) \\ &\quad \times \delta((\Omega+\omega)^2 - (k'+p')^2 - m^2) \end{aligned} \tag{4.104}$$

This shows that independent of the values of the integrals, I and I' (which are finite), the temperature dependent part of the polarization tensor is transverse as would be required by the Ward identities. We note that this analysis can be carried over to $3+1$ dimensions as well and the temperature dependent polarization tensor can be shown to be transverse. This is just a simple example to show that the Ward identities are preserved at finite temperature simply because we have taken care to define the partition functions carefully preserving the BRST invariance. At zero temperature, however, there are several

ways one writes various identities using the inverse of the propagators. At finite temperature, on the other hand, the inverse of the propagator is not unique (simply because the temperature dependent part is annihilated by, say, $(p^2 - m^2)$ for a scalar field. Therefore, one should be careful about what one means by the Ward identities. The correct identities are the ones that follow from the 1PI generating functional as discussed in eq. (4.45).

4.8 References

Becchi, C, A. Rouet and **R. Stora**, Comm. Math. Phys. **42** (1975) 127; Ann. Phys. **98** (1976) 287.

Bernard, C, Phys. Rev. **D9** (1974) 3312.

Das, A and **M. Hott**, Mod. Phys. Lett. **A9** (1994) 3383.

Faddeev, L and **N. Popov**, Phys. Lett. **25B** (1967) 29.

Hata, H and **T. Kugo**, Phys. Rev. **D21** (1980) 3333.

't Hooft, G, Nuc. Phys. **B33** (1971) 173; *ibid* **B35** (1971) 167.

Kugo, T and **I. Ojima**, Prog. Theo. Phys. **60** (1978) 1869; Prog. Theo. Phys. Suppl. No. **66** (1979).

Ojima, I, Ann. Phys. **137** (1981) 1.

Chapter 5

Cutting Rules at $T \neq 0$

In dealing with quantum field theories, often we are interested in calculating the imaginary part of a Feynman amplitude. For example, at zero temperature, the imaginary part of a scattering amplitude is related to the scattering cross section through the unitarity relation. Namely, let

$$S = 1 + iT \tag{5.1}$$

where we have expressed the S-matrix in terms of the pure scattering matrix, T. Formal unitarity of the S-matrix, then, implies that

$$\begin{aligned} SS^\dagger &= 1 \\ \text{or, } -i(T - T^\dagger) &= TT^\dagger \\ \text{or, } -i\langle f|(T - T^\dagger)|i\rangle &= \langle f|TT^\dagger|i\rangle = \sum_p \langle f|T|p\rangle\langle p|T^\dagger|i\rangle \end{aligned} \tag{5.2}$$

The left hand side of this equation simply gives the imaginary part of the scattering matrix taken between an initial state $|i\rangle$ and a final state $|f\rangle$ while the right hand side expresses it as a product of transition amplitudes summed over all possible intermediate states. Similarly, at finite temperature, the imaginary parts of correlation functions can be related to various transport coefficients.

Thus, we see that the calculation of imaginary parts of amplitudes is quite important both at zero temperature as well as at finite temperature. But as is clear from any direct computation, even at zero temperature, such a calculation is quite nontrivial. The cutting rules or Cutkosky rules as they are called, give a simple and systematic way for calculating the imaginary part of a Feynman amplitude which further makes the connection with unitarity manifest. They express the imaginary part of an n-loop amplitude in terms of physical amplitudes of lower order (namely, $(n-1)$-loop or lower). It is in this sense that they are quite useful. In the next section, we will discuss the cutting rules at zero temperature which we will subsequently generalize to finite temperature.

5.1 Zero Temperature

The discussion of cutting rules is quite straightforward in the diagrammatic language and that is what we will follow in this chapter. In calculating the imaginary part of a scattering amplitude, it is quite clear that the imaginary parts can arise from two distinct sources. First, the vertices are defined from $i\mathcal{L}$ where $\mathcal{L}$ represents

the Lagrangian density of the theory. We see that if the interactions are Hermitian, then, the vertices are purely imaginary. Second, the Feynman propagators are complex because of the $i\epsilon$ prescription (as well as the factor of "i" with which a propagator is defined) and will contribute to the imaginary part of an amplitude. It is clear that since the imaginary part of an amplitude can be obtained from the amplitude and its complex conjugate, we can give it a diagrammatic representation if we enlarge our theory to include also the complex conjugate vertices as well as the complex conjugate propagators of our original theory. We recognize that, for Hermitian interactions, complex conjugating the vertex is simply changing the sign of the vertex. However, the complex conjugate of the propagator is not so simple.

For simplicity, let us consider a scalar field theory although the discussion can be carried over to any other theory. The Feynman propagator is the time ordered two point Greens function defined to be

$$iG_F(x) = \langle 0|T(\phi(x)\phi(0))|0\rangle$$

(The propagator is normally denoted by iG. However, for simplicity of notation, we will represent the propagators, only in this chapter, by G. The formulae of chapters 2 and 3 can, therefore, be compared with those of the present chapter keeping in mind the identification $iG \to G$.) This has the explicit representation, in 4-dimensions,

$$\begin{aligned} G_F(x) &= \theta(x^0)G^+(x) + \theta(-x^0)G^-(x) \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot x} \end{aligned} \tag{5.3}$$

where we are assuming that the scalar particles have a mass m. Here $G^{\pm}(x)$ are the usual positive and negative frequency propagators in terms of which all other Greens functions can be expressed and have the explicit representations

$$G^{\pm}(x) = \int \frac{d^4p}{(2\pi)^4}\, 2\pi\theta(\pm p^0)\delta(p^2 - m^2)\, e^{-ip\cdot x} \tag{5.4}$$

The positive and the negative frequency propagators have support only on the mass shell while the Feynman propagator is an off shell Greens function. Parenthetically, we note here that the Feynman propagator of eq. (5.3) also has the Källen-Lehmann spectral representation given by

$$G_F(x) = i\int_0^{\infty} ds \int \frac{d^4p}{(2\pi)^4}\, \frac{\rho(s,p)}{p^2 - s + i\epsilon}\, e^{-ip\cdot x}$$

where the spectral function, ρ, at zero temperature, has the explicit form

$$\rho(s,p) = \rho(s) = \delta(s - m^2)$$

The coefficients of $\theta(\pm x^0)$ of a Greens function can be easily read out once we know its spectral decomposition.

As is clear from eq. (5.4), the positive and negative frequency propagators are not even functions. Rather, they satisfy the following symmetry properties.

$$G^{\pm}(-x) = G^{\mp}(x) = (G^{\pm}(x))^* \tag{5.5}$$

They are, in fact, complex conjugates of each other. On the other hand, we see, from eq. (5.3), that while the Feynman propagator is not a real function, it is an even function, namely,

$$G_F(-x) = G_F(x) \tag{5.6}$$

We also note that if we were to define an anti-time ordered propagator, it will have the form

$$\begin{aligned} \tilde{G}(x) &= \theta(x^0)G^-(x) + \theta(-x^0)G^+(x) \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 - m^2 - i\epsilon} e^{-ip\cdot x} \end{aligned} \tag{5.7}$$

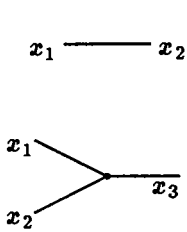
It is immediately clear from eqs. (5.3), (5.5), (5.7) that the anti-time ordered propagator is the complex conjugate of the Feynman propagator, namely,

$$\tilde{G}(x) = (G_F(x))^* \tag{5.8}$$

We also note here that not all of these propagators are independent. In fact, from the definitions in eqs. (5.3) and (5.7), as well as the properties of the θ-functions, we see that

$$\begin{aligned} G_F(x) + \tilde{G}(x) &= (\theta(x^0) + \theta(-x^0))G^+(x) + (\theta(x^0) + \theta(-x^0))G^-(x) \\ &= G^+(x) + G^-(x) \end{aligned} \tag{5.9}$$

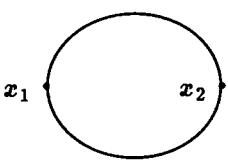
Given a theory, we can define the Feynman rules associated with the theory. Any Feynman amplitude, then, simply consists of a number of vertices and propagators. In the coordinate representation, therefore, the integrand of the amplitude can be written as a function of the coordinates of the vertices (that is, before integrating over the internal vertices). Thus, for example, for the scalar theory with ϕ^3 interactions, we can write the Feynman rules to be



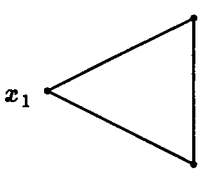
$$= G_F(x_1 - x_2)$$

$$= -ig \tag{5.10}$$

from which we can obtain a representative set of Feynman amplitudes to be



$$= F(x_1, x_2) = (-ig)^2 G_F(x_1 - x_2) G_F(x_2 - x_1) \tag{5.11}$$



$$= F(x_1, x_2, x_3)$$

$$= (-ig)^3 G_F(x_1 - x_2) G_F(x_2 - x_3) G_F(x_3 - x_1)$$

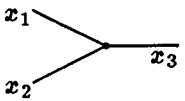
Let us, next, enlarge our theory such that the Feynman rules of the enlarged theory are given by

$$x_1 \text{ ——— } x_2 = G_F(x_1 - x_2)$$

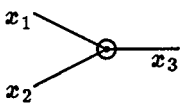
$$x_1 \text{ ———○ } x_2 = G^-(x_1 - x_2) = G^+(x_2 - x_1)$$

$$x_1 \text{ ○——— } x_2 = G^+(x_1 - x_2) = G^-(x_2 - x_1)$$

$$x_1 \circ\!\!-\!\!-\!\!\circ x_2 \;=\; G_F^*(x_1 - x_2) = \tilde{G}_F(x_1 - x_2)$$



$$= -ig$$



$$= ig \tag{5.12}$$

In the momentum space the propagators have the following form

$$\underset{p}{\text{———}} \;=\; \frac{i}{p^2 - m^2 + i\epsilon}$$

$$\underset{p}{\text{———}\circ} \;=\; 2\pi\theta(-p^0)\delta(p^2 - m^2)$$

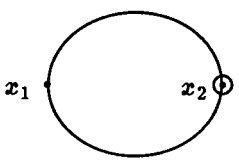
$$\underset{p}{\circ\text{———}} \;=\; 2\pi\theta(p^0)\delta(p^2 - m^2)$$

$$\underset{p}{\circ\text{———}\circ} \;=\; -\frac{i}{p^2 - m^2 - i\epsilon} \tag{5.13}$$

where we note that the direction of the momentum flow is quite important when one of the ends of a propagator is circled because, as we have observed earlier, such propagators correspond to the positive/negative frequency propagators which are not even functions.

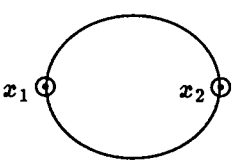
The important thing to note from the enlarged set of Feynman rules is the fact that the circled vertex is merely the complex conjugate of the original vertex and similarly, the doubly circled propagator (or the anti-time ordered propagator) is the complex conjugate of the Feynman propagator of the ϕ^3 theory. It is clear, therefore, that the new Feynman rules of the enlarged theory also contain the complex conjugates of the vertex as well as the propagator of the original

theory. Furthermore, with these new Feynman rules, we can draw new kinds of Feynman diagrams where some of the vertices may be circled. The integrand of a Feynman amplitude, in this new theory, can still be represented as a function of the coordinates of the vertices and we will represent a circled vertex merely with the corresponding coordinate underlined. With this, then, some of the new amplitudes can be written as



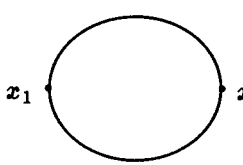
$$= F(x_1, \underline{x_2})$$

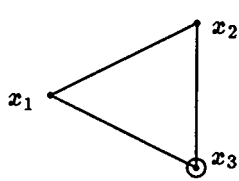
$$= (-ig)(ig)G^-(x_1 - x_2)G^-(x_1 - x_2)$$



$$= F(\underline{x_1}, \underline{x_2}) = (ig)^2 G_F^*(x_1 - x_2)G_F^*(x_1 - x_2)$$

$$= \left(\right)^*$$





$$= F(x_1, x_2, \underline{x_3}) = (-ig)^2(ig)$$
$$\times G_F(x_1 - x_2)G^-(x_2 - x_3)G^-(x_1 - x_3)$$

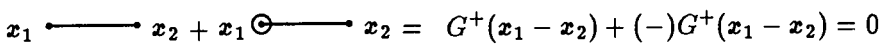
(5.14)

It is now clear that an amplitude where all the vertices are circled is nothing other than the complex conjugate of the amplitude without any circling (namely, of the original theory). This is because in such a case, all the vertices as well as the propagators are the complex conjugates of the original diagram. Algebraically, we can write

$$F(\underline{x_1}, \underline{x_2}, \cdots, \underline{x_n}) = (F(x_1, x_2, \cdots, x_n))^* \tag{5.15}$$

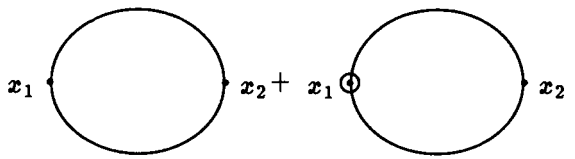
This, therefore, gives us a diagrammatic method for constructing the complex conjugate and, therefore, the imaginary part of a Feynman amplitude of the original theory. However, it is not yet in a simple form which is convenient for computations.

The graphs with circled vertices satisfy many identities simply because the propagators satisfy various identities. Let us note here one that will be useful in the description of the cutting rules. Let us first consider a two point amplitude with coordinates x_1 and x_2 where we assume that $x_1^0 > x_2^0$. Then, it follows that

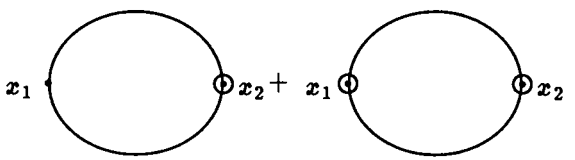


(5.16)

(5.17)

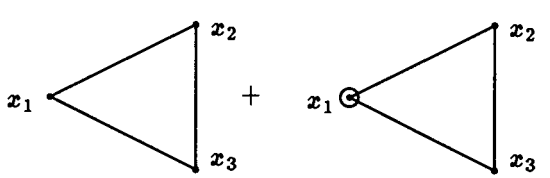


$$= (-ig)^2(G^+(x_1 - x_2))^2 + (-ig)(ig)(G^+(x_1 - x_2))^2 = 0 \qquad (5.18)$$

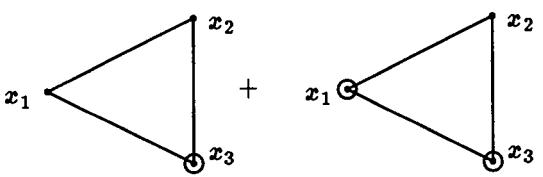


$$= (-ig)(ig)(G^-(x_1 - x_2))^2 + (ig)^2(G^-(x_1 - x_2))^2 = 0 \qquad (5.19)$$

In deriving these relations, we have used the fact that a circled vertex gives an extra $(-)$ sign and that for $x_1^0 > x_2^0$, $G_F(x_1 - x_2) = G^+(x_1 - x_2)$ and $\tilde{G}(x_1 - x_2) = G^-(x_1 - x_2)$. Next, let us consider a three point function with coordinates x_1, x_2 and x_3 and let us assume that x_1^0 represents the largest of the three time coordinates. Then, it is easy to check that

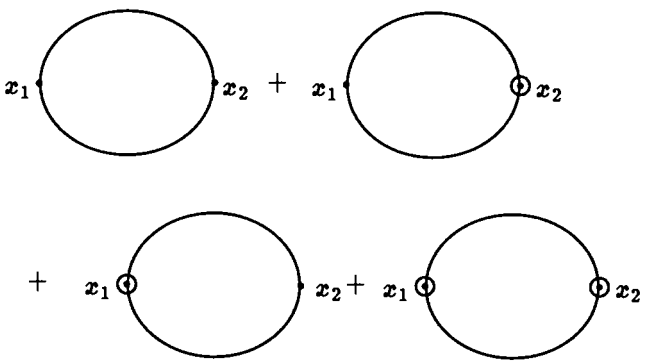


$$= (-ig)^3 [G^+(x_1 - x_3)\, G_F(x_2 - x_3)\, G^+(x_1 - x_2) + (-)\, G^+(x_1 - x_3)\, G_F(x_2 - x_3)\, G^+(x_1 - x_2)] = 0 \quad (5.20)$$



$$= (-ig)^3[(-)\, G^-(x_1 - x_3)\, G^+(x_3 - x_2)\, G^+(x_1 - x_2) + (-)^2\, G^-(x_1 - x_3)\, G^+(x_3 - x_2)\, G^+(x_1 - x_2)] = 0 \quad (5.21)$$

In this way, one derives an interesting relation. Namely, in this enlarged theory, if we take an amplitude with the largest time vertex uncircled and add to it the corresponding amplitude with the largest time vertex circled, then, the sum vanishes. This is known as the largest time equation. (Similarly, there is a corresponding smallest time equation also.) An immediate consequence of the largest time equation (or the smallest time equation) is that for a particular amplitude, the sum of graphs with all possible circlings vanish. This can be easily checked, for example, from



$$
\begin{aligned}
&= (-ig)^2[(G_F(x_1-x_2))^2 + (-)(G^-(x_1-x_2))^2 \\
&\quad (-)(G^+(x_1-x_2))^2 + (-)^2(\tilde{G}(x_1-x_2))^2] \\
&= (-ig)^2[\theta(x_1^0-x_2^0)(G^+(x_1-x_2))^2 + \theta(x_2^0-x_1^0)(G^-(x_1-x_2))^2 \\
&\quad -(G^-(x_1-x_2))^2 - (G^+(x_1-x_2))^2 \\
&\quad +\theta(x_1^0-x_2^0)(G^-(x_1-x_2))^2 + \theta(x_2^0-x_1^0)(G^+(x_1-x_2))^2] \\
&= 0 \qquad (5.22)
\end{aligned}
$$

This can be intuitively expected because for a given distribution of the time arguments of a graph, diagrams with the largest time circled and uncircled would cancel pairwise because of the largest time equation. (We note that the number of graphs with all possible circlings corresponding to any given amplitude is even.) It is also worth emphasizing that all these identities hold both in the coordinate space as well as in the momentum space.

Algebraically, therefore, we see that for any Feynman diagram, we can write

$$\sum_{\text{all circlings}} F(x_1, x_2, \cdots, \underline{x_i}, \cdots, x_n) = 0 \tag{5.23}$$

This also implies that

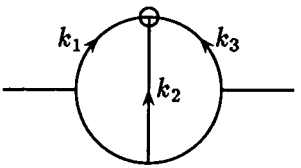
$$F(x_1, x_2, \cdot, x_n) + F(\underline{x_1}, \underline{x_2}, \cdots, \underline{x_n}) = -\sum{}' F(x_1, \cdots, \underline{x_i}, \cdots, x_n) \tag{5.24}$$

Here the prime on the right hand side over the sum stands for all possible circlings except the cases without any and with all circlings. Furthermore, recognizing that a Feynman diagram really gives "i" times the T-matrix element, namely,

$$F \sim iT \tag{5.25}$$

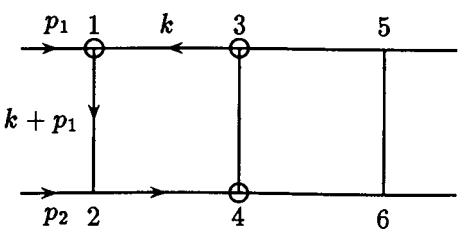
we see that we can relate the left hand side of eq. (5.24) to the imaginary part of a given Feynman amplitude and it can be diagrammatically expressed in terms of graphs with all possible circlings (except no circling and all circlings). The right hand side of eq. (5.24) still involves a large number of graphs and hence not in a simple form for calculations. Besides, it is not yet in a form where the unitarity relation of eq. (5.2) is manifest.

Further simplification of the right hand side of eq. (5.24) comes from the following observations. Let us consider the following self-energy graph with an isolated internal vertex which is circled. In this case, we see quite easily, from the definition of the propagators in eq. (5.13) that



$$\sim G^-(k_1)G^-(k_2)G^-(k_3) \sim \theta(-k_1^0)\theta(-k_2^0)\theta(-k_3^0) = 0 \qquad (5.26)$$

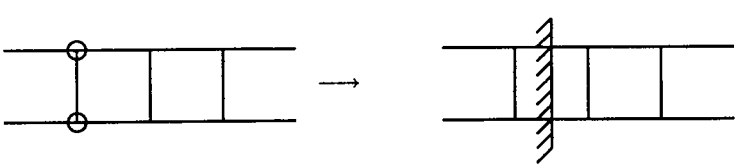
where the last equality follows because, by energy-momentum conservation, $k_1 + k_2 + k_3 = 0$. This simple analysis shows that any Feynman diagram with an isolated circled internal vertex vanishes. This reduces the number of diagrams to be considered on the right hand side of eq. (5.24). In a completely parallel manner, we can show that energy-momentum conservation makes Feynman diagrams with an isolated uncircled internal vertex also vanish (In fact, this follows trivially because a diagram with an isolated uncircled vertex is the complex conjugate of a diagram with an isolated circled vertex.). Furthermore, if we assume that particles are incoming from the left and are going out at the right – that is, if we assume that energy is flowing to the right – then, it is straightforward to show that we cannot have any uncircled vertex attached to the external lines on the left surrounded by circled vertices. Thus, for example, the following graph vanishes

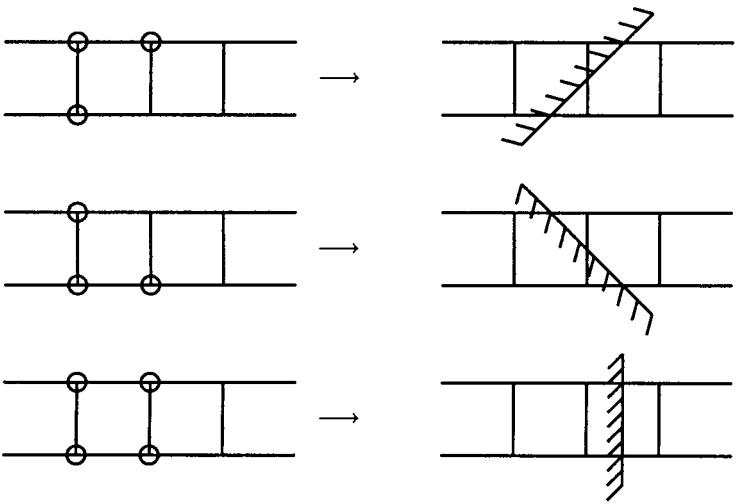


$$\sim \theta(k^0 + p_1^0)\theta(-k^0 - p_1^0 - p_2^0) = 0 \qquad (5.27)$$

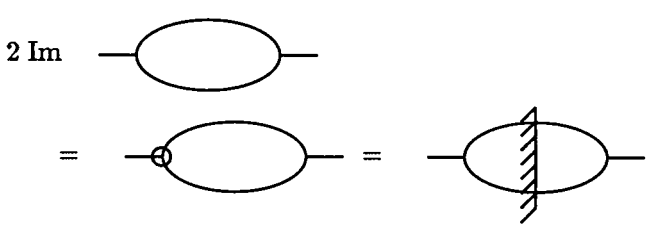
because of energy conservation at vertex 2. In fact, a systematic analysis shows that the only diagrams that can give nonvanishing contribution to the right hand side of eq. (5.24) are the ones where the circled and the uncircled vertices form connected regions, connected with the external lines with no isolated islands (of either circled or uncircled type). Furthermore, with our convention of representing the incoming particles coming in at the left and leaving at the right, diagrams with only circled vertices attached to the external lines at the left and uncircled vertices attached to the external lines at the right can give nonvanishing contribution. (With the opposite convention, namely, particles coming in at the right and going out at the left, the nonvanishing diagrams would merely correspond to the transposed ones.)

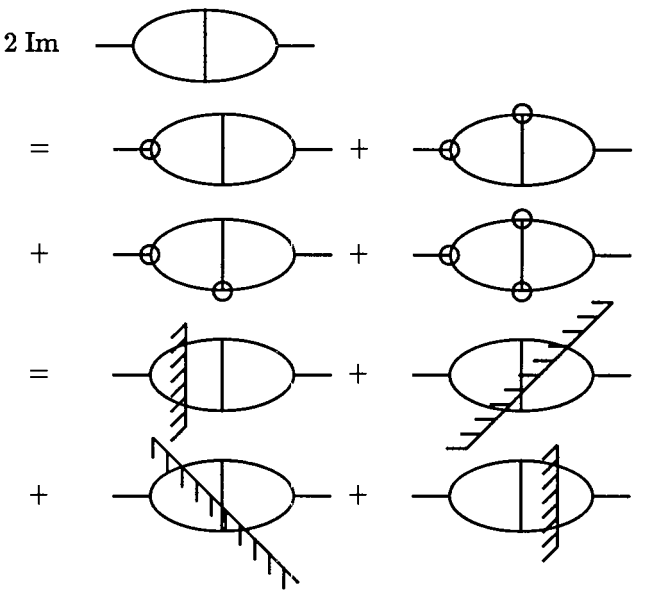
With these properties, we realize that a large number of graphs on the right hand side of eq. (5.24) vanish and that the ones that are nontrivial can be given a cutting description since the circled and the uncircled vertices form connected regions. Thus, for example, we can represent



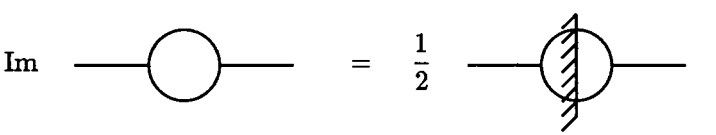


where any vertex in the shadow region is understood to be a circled vertex. Since all the vertices and the propagators in the shadow region correspond to the complex conjugates of vertices and propagators of the original theory, the part of the diagram in the shadow can be thought of as the complex conjugate of a matrix element of the original theory. As a consequence, we can write the imaginary part of any Feynman amplitude in terms of cut diagrams which will also lead to a natural interpretation in terms of unitarity (see eq. (5.2)). Thus, for example, we note that





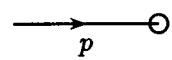
In general, we can write (for a two point function, in this case)



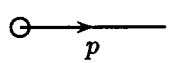
(5.28)

These are the cutting rules at zero temperature which say that the imaginary part of a given Feynman diagram is given by the sum of all possible cuttings of the diagram which separate the circled and the uncircled vertices as well as the external lines such that there are no isolated islands of circled/uncircled internal vertices not connected to the external lines. We note that a cut propagator is an on-shell

propagator (since it connects a circled and an uncircled vertex) and, therefore, the cutting rules express the imaginary part of a Feynman amplitude in terms of physical, on-shell amplitudes of lower order. We also note here that crucial to the cutting description of the imaginary part is the fact that a diagram with an isolated circled/uncircled internal vertex vanished. This follows because at zero temperature, a propagator with one end circled and the other uncircled has contribution only from positive/negative energy.

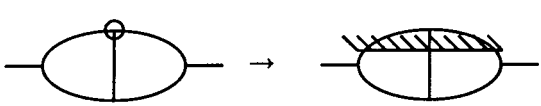


$$= 2\pi\theta(-p^0)\delta(p^2 - m^2)$$



$$= 2\pi\theta(p^0)\delta(p^2 - m^2) \qquad (5.29)$$

As we will see later, this is no longer true at finite temperature. In fact, let us note here that if such diagrams, with an isolated circled/uncircled internal vertex, did not vanish, then, the imaginary part of an amplitude cannot have a cutting description in terms of physical amplitudes which is evident from the following simple self-energy diagram.



(5.30)

5.2 Finite Temperature

We have already seen, in chapter 1, in the imaginary time formalism, that the one-loop self-energy at finite temperature exhibits a cutting structure as long as we allow for additional channels of reaction present in a medium (see eq. (1.64) as well as the discussion following that). It is natural, therefore, to ask whether the notion of cutting generalizes to all orders at finite temperature. As we have noted earlier, if this were true, calculation of imaginary parts of amplitudes which are related to transport coefficients would simplify enormously.

The question of cutting rules is best described in the real time formalism and we will use the closed time path formalism for our discussions. (Our arguments can be extended to thermofield dynamics with appropriate modifications.) The real time formalism at finite temperature differs from the zero temperature field theory mainly in two respects. First, there is a doubling of the degrees of freedom which leads to a 2×2 matrix structure for the propagator and the presence of two kinds of vertices with opposite sign. Second, the fields obey a periodic/anti-periodic boundary condition with the Greens functions satisfying the KMS conditions (see eqs. (1.6) or (3.84)). Thus, for example, for a ϕ^3 theory, the propagator in the momentum space, as we have seen in eq. (2.36), are given by (Remember that, in this chapter, we are using $iG \to G$.)

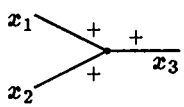
$$
\begin{aligned}
G_{++}(p) &= \frac{i}{p^2 - m^2 + i\epsilon} + 2\pi n(|p^0|)\delta(p^2 - m^2) \\
G_{+-}(p) &= 2\pi(n(|p^0|) + \theta(-p^0))\delta(p^2 - m^2)
\end{aligned}
$$

$$
\begin{aligned}
G_{-+}(p) &= 2\pi(n(|p^0|) + \theta(p^0))\delta(p^2 - m^2) \\
G_{--}(p) &= -\frac{i}{p^2 - m^2 - i\epsilon} + 2\pi n(|p^0|)\delta(p^2 - m^2)
\end{aligned}
\tag{5.31}
$$

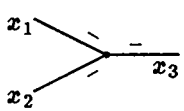
where the boson distribution function is given by

$$
n(|p^0|) = \frac{1}{e^{\beta|p^0|} - 1}
$$

The two kinds of vertices, similarly, are given by



$$= -ig \tag{5.32}$$



$$= ig$$

It is worth noting here, from the structure of the propagators in eq. (5.31), that

$$
\begin{aligned}
G_{--}(p) &= (G_{++}(p))^* \\
G_{++}(p) + G_{--}(p) &= G_{+-}(p) + G_{-+}(p)
\end{aligned}
$$

Namely, these identities involving the thermal propagators are very similar to the zero temperature ones in eqs. (5.8) and (5.9) respectively.

Given the form of the momentum space propagators in eq. (5.31), we can write the spectral representation of the matrix propagators. Because of the δ-functions in eq. (5.31), the propagators (unlike

the zero temperature case) have contributions from the entire complex energy plane and consequently, the spectral representation takes the form

$$G_{ab}(x) = i\int_0^\infty ds \int \frac{d^4p}{(2\pi)^4}\left[\frac{\rho_{ab}(s,p)}{p^2 - s + i\epsilon} + \frac{\tilde{\rho}_{ab}(s,p)}{p^2 - s - i\epsilon}\right] e^{-ip\cdot x} \quad (5.33)$$

where the spectral functions ρ_{ab} and $\tilde{\rho}_{ab}$ can be read out from eq. (5.31) to be

$$\begin{aligned}
\rho_{++}(s,p) &= (1 + n(|p^0|))\delta(s - m^2) = -\tilde{\rho}_{--}(s,p) \\
\rho_{--}(s,p) &= n(|p^0|)\delta(s - m^2) = -\tilde{\rho}_{++}(s,p) \\
\rho_{+-}(s,p) &= (n(|p^0|) + \theta(-p^0))\delta(s - m^2) = -\tilde{\rho}_{+-}(s,p) \\
\rho_{-+}(s,p) &= (n(|p^0|) + \theta(p^0))\delta(s - m^2) = -\tilde{\rho}_{-+}(s,p) \quad (5.34)
\end{aligned}$$

Given the spectral representations in eqs. (5.33) and (5.34), we can now have the decomposition of the propagators as

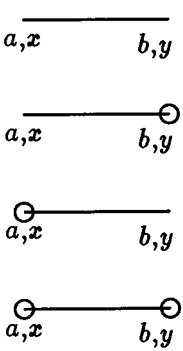
$$G_{ab}(x) = \theta(x^0)G^+_{ab}(x) + \theta(-x^0)G^-_{ab}(x) \quad (5.35)$$

where we have

$$\begin{aligned}
G^\pm_{ab}(x) &= \int_0^\infty ds \int \frac{d^4p}{(2\pi)^4}\, 2\pi\delta(p^2 - m^2)\, e^{-ip\cdot x} \\
&\times \left[\theta(\pm p^0)\rho_{ab}(s,p) - \theta(\mp p^0)\tilde{\rho}_{ab}(s,p)\right] \quad (5.36)
\end{aligned}$$

Following the discussion of the zero temperature case, we can now introduce the circled propagators for our enlarged theory

at finite temperature. They are given by $(a, b = \pm)$



$$= G_{ab}(x-y)$$

$$= G^{-}_{ab}(x-y)$$

$$= G^{+}_{ab}(x-y)$$

$$= \tilde{G}_{ab}(x-y) = (G_{ab}(x-y))^{*} \qquad (5.37)$$

The important point to note here is that, like at zero temperature, all the propagators can be expressed in terms of two fundamental functions. In fact, from the structure of the spectral functions in eq. (5.34), we can easily show that eq. (5.36) gives

$$\begin{aligned} G^{+}_{+-}(x) &= G^{-}_{+-}(x) = G_{+-}(x) \\ G^{+}_{-+}(x) &= G^{-}_{-+}(x) = G_{-+}(x) \\ G^{+}_{++}(x) &= G^{-}_{--}(x) = G_{-+}(x) \\ G^{-}_{++}(x) &= G^{+}_{--}(x) = G_{+-}(x) \end{aligned} \qquad (5.38)$$

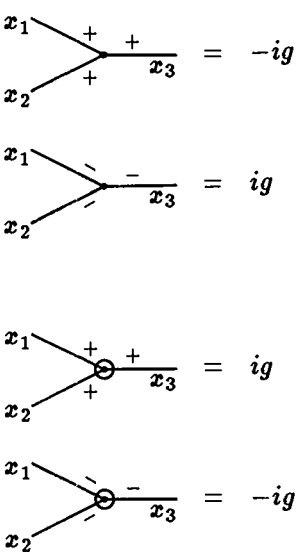
From the explicit forms of these Greens functions in momentum space, given in eq. (5.31), we note that

$$G_{+-}(p) = 2\pi(\theta(p^0)n(|p^0|) + \theta(-p^0)(1+n(|p^0|)))\delta(p^2-m^2) \qquad (5.39)$$

$$G_{-+}(p) = 2\pi(\theta(p^0)(1+n(|p^0|)) + \theta(-p^0)n(|p^0|))\delta(p^2-m^2)$$

As a result, a propagator with one end circled and the other uncircled, while on-shell, would contain both positive and negative energy contributions unlike at zero temperature. This has the immediate consequence that unlike at zero temperature, at finite temperature, isolated circled vertices will no longer vanish (see eq. (5.29) and the discussion there). As we had argued earlier, vanishing of such diagrams was quite essential for the cutting description and, therefore, this raises a major obstacle in generalizing the cutting rules to finite temperature.

To see how exactly the cutting rules generalize to finite temperature, we note that in this enlarged theory at finite temperature, we would have four possible vertices

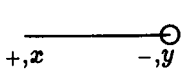


(5.40)

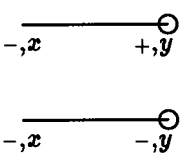
With these, one can derive, once again like at zero temperature, the largest time (smallest time) equation. This would again lead to the result that for a given amplitude with a given distribution of thermal vertices, the sum over all possible circlings must vanish which, in turn, would give the imaginary part of a diagram with a given distribution of thermal vertices as in eq. (5.24). As we have argued before, imaginary parts of these individual diagrams cannot have a cutting description at finite temperature simply because isolated circled vertices no longer vanish. (Incidentally, such nontrivial graphs arise only at two loops or higher order.) However, at finite temperature, a physical amplitude is obtained by summing over all possible thermal indices of the internal vertices (mainly because of the matrix structure of the theory). Therefore, although at finite temperature, non-cuttable diagrams occur graph by graph, it is possible that when summed over the thermal indices of the internal vertices, all such diagrams would cancel giving a cutting description to the complete, physical amplitude. This is exactly what happens as we will show next.

To show the cancellation of the non-cuttable graphs, let us start with the following interesting observations. First, let us explicitly write down the forms of the propagators where one end is circled (This follows from eqs. (5.37) and (5.38).)

$$= G_{+-}(x-y)$$



$$= G_{+-}(x-y)$$



$$= G_{-+}(x-y)$$

$$= G_{-+}(x-y) \tag{5.41}$$

It is immediately clear that the form of the propagators, when one end is circled, is independent of the thermal index of the circled vertex and is completely determined by the thermal index of the uncircled vertex. This, as we will see, is crucial to proving various cancellations. The other fact that is quite important is that, at finite temperature, Greens functions obey the KMS condition (see eqs. (1.6) or (3.84)) which implies that (which can also be explicitly checked from eq. (2.29))

$$G_{+-}(t,\vec{x}) = G_{-+}(t-i\beta,\vec{x}) \tag{5.42}$$

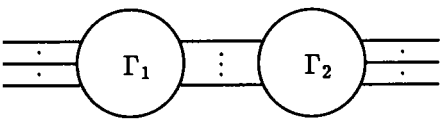
In the momentum space, this implies that

$$\begin{aligned} G_{+-}(p) &= e^{-\beta p^0}\, G_{-+}(p) \\ \text{or, } G_{-+}(p) &= e^{\beta p^0}\, G_{+-}(p) \end{aligned} \tag{5.43}$$

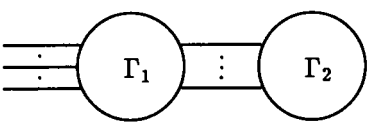
This can, of course, be checked explicitly from the forms of the propagators given in eq. (5.31) and the properties of the distribution function.

The next important thing to recognize is that any diagram in this enlarged theory at finite temperatures is just a collection of circled and uncircled vertices (of given distribution of thermal indices) and can be drawn in one of the following three topologically distinct

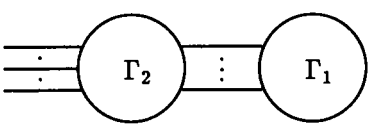
categories. (There can also be class I diagrams with the blobs Γ_1 and Γ_2 interchanged.)



Class I

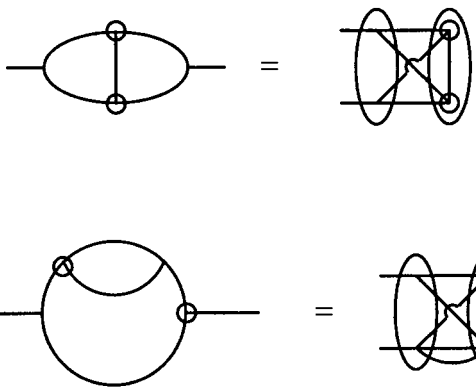


Class II



Class III

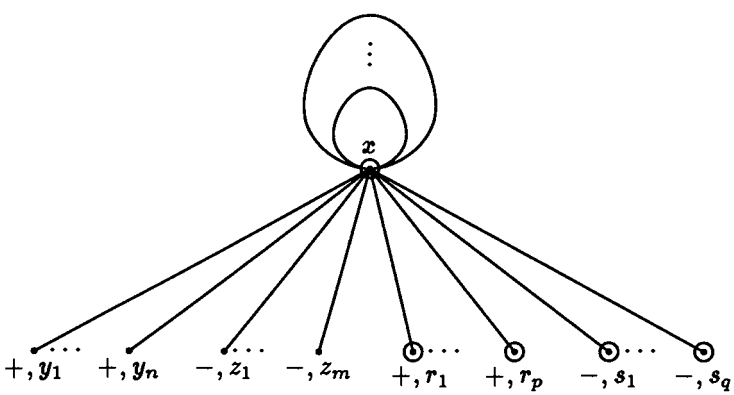
Here we assume that the blob, Γ_1, contains only circled vertices whereas the blob, Γ_2, contains only uncircled vertices. The vertices may be connected, between the blobs, in a highly nontrivial manner and that not all vertices in a blob may be connected to each other. Thus, for example, we can write



It is clear that the second and the third topological classes of diagrams cannot be given a cutting description because any cutting would give rise to an island of isolated circled/uncircled vertices not connected to the external lines. Parts of the first topological class of diagrams where cuttings can produce isolated islands of disconnected circled/uncircled vertices also cannot be given a cutting description. Therefore, if we can prove that these classes of diagrams vanish when summed over the internal thermal indices, we would have shown that the cutting rules of zero temperature generalize to finite temperature as well, for the whole amplitude – not graph by graph.

Let us now consider the class of diagrams where there is an isolated set of circled internal vertices, not connected to the external lines. This could be, for example, diagrams of the class III or those of class I where there exists a disconnected set of internal circled vertices. Clearly, since these circled vertices are all internal, their thermal index has to be summed over in the calculation of any amplitude. Let us label

the coordinate of the vertex among these circled ones with the smallest time to be x. This vertex, of course, may be connected to uncircled vertices of the type + at coordinates $y_1, y_2, \cdots, y_n$ as well as to uncircled vertices of the type − at coordinates $z_1, z_2, \cdots, z_m$. In addition, the circled vertex at x may also be connected to the circled vertices of the cluster of the type + at $r_1, r_2, \cdots, r_p$ and of the type − at $s_1, s_2, \cdots, s_q$. The vertex x may, in general, also be connected to itself l-times. Let us next sum over the thermal index of the circled coordinate at x, for such a general graph, with all other thermal indices fixed.



In evaluating this sum, the crucial thing to note is the observation we made earlier, namely, the propagator connecting a circled and an uncircled vertex does not depend on the thermal index of the circled vertex. As a result, in the sum, the propagators connecting x to the uncircled vertices factorize (since we have fixed the thermal indices of the uncircled vertices) and we have

$$\sum_{a=\pm} \text{diagram} =$$

$$R G_{-+}(x-y_1)\cdots G_{-+}(x-y_n) G_{+-}(x-z_1)\cdots G_{+-}(x-z_m)$$

$$\times\Big[(G_{--}(0))^l G_{--}(x-r_1)\cdot\cdot G_{--}(x-r_p) G_{+-}(x-s_1)\cdot\cdot G_{+-}(x-s_q)$$

$$-(G_{++}(0))^l G_{-+}(x-r_1)\cdot\cdot G_{-+}(x-r_p) G_{++}(x-s_1)\cdot\cdot G_{++}(x-s_q)\Big] \tag{5.44}$$

Here R stands for the rest of the diagram whose actual form is not of interest except the fact that it remains the same for the two values of the thermal index at x, $a = \pm$. We can now use various properties of the Greens functions such as

$$G_{++}(0) = G_{--}(0)$$

and furthermore, with x^0 the smallest time among the circled vertices, we have (see eqs. (5.35) and (5.38))

$$\begin{aligned} G_{++}(x-s_i) &= G_{+-}(x-s_i) \\ G_{--}(x-r_i) &= G_{+-}(x-r_i) \end{aligned}$$

which leads to the result that

$$\sum_{a=\pm} \text{diagram} = R(G_{++}(0))^l$$

$$\times G_{-+}(x-y_1)\cdots G_{-+}(x-y_n) G_{+-}(x-z_1)\cdots G_{+-}(x-z_m)$$

$$\times\,[G_{-+}(x-r_1)\cdots G_{-+}(x-r_p) G_{+-}(x-s_1)\cdots G_{+-}(x-s_q)$$

$$-\,G_{-+}(x-r_1)\cdots G_{-+}(x-r_p) G_{+-}(x-s_1)\cdots G_{+-}(x-s_q)]$$

$$= 0 \tag{5.45}$$

This result is quite important, for it says that diagrams of class III as well as those of class I with an isolated internal island of circled vertices vanish when summed over the internal thermal indices. It is worth emphasizing here that there were two essential ingredients in proving this result. First, as we have noted before, the propagators where one end is circled and the other is not, does not depend on the thermal index of the circled vertex and, as such, such propagators remain the same when we sum over the thermal index of the circled vertex and factorize. The second crucial point is that we are able to choose a vertex with the smallest time among the circled vertices whose thermal index is being summed. We note that, since the circled vertices are internal vertices, we have to integrate over their coordinates as well as sum over their thermal indices in evaluating the amplitude and, consequently, among the circled vertices, there will always be one with the smallest time whose thermal index is being summed. In contrast, if any of these vertices were connected to the external lines, then, for some distribution of the time variables of the circled vertices, the time coordinate of the external circled vertex will be the smallest. But being an external vertex, such a vertex is not summed over the thermal index and hence our result would not apply to make such diagrams vanish.

From the result that we have just proved, it would seem that we can also prove the complementary result trivially, namely, that diagrams with an isolated island of uncircled internal vertices (not connected to external lines) also vanish when summed over the thermal indices of the internal vertices. But, in fact, the previous result does not automatically generalize if we interchange the circled and uncircled

vertices. This is because, in the present case, the internal vertices are uncircled and we are summing over their thermal indices. As we have noted before, propagators connecting a circled and an uncircled vertex are completely determined by the thermal index of the uncircled vertex. As a result, when we sum over the thermal index of an uncircled vertex, such propagators will no longer remain the same and will not factorize, unlike in the earlier case, and the proof will not go through. To understand the structure of such diagrams and show that they vanish when summed over internal thermal indices, we somehow need to factor out the propagators connecting a circled and an uncircled vertex.

The factorization of these propagators can be more easily seen in the momentum space as follows. Let us consider a class of diagrams where there is an isolated island of internal uncircled vertices. This could, for example, be diagrams of class II or those of class I where there is a disconnected island of uncircled internal vertices. Let $k_1, k_2, \cdots, k_n$ represent the momenta flowing from Γ_1 to Γ_2 along the propagators connecting the two blobs. Since these propagators necessarily have one end circled and the other uncircled, their form is completely determined by the thermal indices of the uncircled vertices, namely, those in Γ_2. Γ_2, of course, will have other uncircled vertices not connected to the blob Γ_1. Let us consider a fixed distribution of thermal indices for such uncircled vertices and sum over the thermal indices of those connected to Γ_1. In such a case, we can write

$$\Sigma(p) = \Gamma(p, k_1, \cdots, k_n) \times \sum_{\alpha_1, \cdots, \alpha_n = \pm} G_{-\alpha_1 \alpha_1}(k_1) \cdots G_{-\alpha_n \alpha_n}(k_n)\, \Lambda_{\alpha_1, \cdots, \alpha_n}(k_1, \cdots, k_n) \tag{5.46}$$

Here p represents the external momentum entering the diagram through Γ_1 so that

$$p = k_1 + k_2 + \cdots + k_n$$

Furthermore, $\Gamma(p, k_1, \cdots, k_n)$ represents the part of the amplitude in Γ_1 containing only doubly circled propagators while $\Lambda_{\alpha_1,\cdots,\alpha_n}(k_1, \cdots, k_n)$ the part of the amplitude in Γ_2 that contains only uncircled vertices. Let us also assume, for simplicity, that all the factors of i coming from the vertices are contained in $\Gamma(p, k_1, \cdots, k_n)$.

Let us now concentrate on the sum over the i-th thermal index on the right side of eq. (5.46). This would give

$$\begin{aligned}\Sigma(p) &= \Gamma(p, k_1, \cdots, k_n) \\ &\times \sum \{[G_{-+}(k_i)\Lambda_{\alpha_1,\cdots,+,\cdots,\alpha_n} + G_{+-}(k_i)\Lambda_{\alpha_1,\cdots,-,\cdots,\alpha_n}] \\ &\times G_{-\alpha_1\alpha_1}(k_1)\cdots G_{-\alpha_n\alpha_n}(k_n)\} \qquad (5.47)\end{aligned}$$

Here the sum, on the right hand side, is over all vertices except the i-th one. This makes it very clear now that when we sum over the thermal index of an uncircled vertex connected with a circled vertex, the propagators are different. However, we can use the KMS relation of eq. (5.43) to factorize the propagator as

$$\begin{aligned}\Sigma(p) &= \Gamma(p, k_1, \cdots, k_n)G_{+-}(k_i)\sum \{[e^{\beta k_i^0}\Lambda_{\alpha_1,\cdots,+,\cdots,\alpha_n} + \Lambda_{\alpha_1,\cdots,-,\cdots,\alpha_n}] \\ &\times G_{-\alpha_1\alpha_1}(k_1)\cdots G_{-\alpha_n\alpha_n}\} \qquad (5.48)\end{aligned}$$

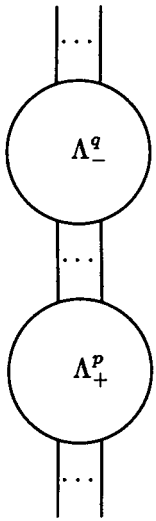
The sum over all the thermal indices can now be done in a similar manner leading to the result that

$$\Sigma(p) = \Gamma(p, k_1, \cdots, k_n) G_{+-}(k_1) \cdots G_{+-}(k_n) \times \sum_{\alpha_1, \cdots, \alpha_n = \pm} \Lambda_{\alpha_1, \cdots, \alpha_n} e^{\sum_{j=1}^{n} \frac{a_j + 1}{2} \beta k_j^0} \tag{5.49}$$

where

$$a_j = \begin{cases} +1 & \text{for} \quad \alpha_j = + \\ -1 & \text{for} \quad \alpha_j = - \end{cases}$$

The important thing to note here is that all the propagators connecting a circled and an uncircled vertex have now been factored out and in the sum that remains, there is an appropriate exponential factor for every index α_j that is positive. Therefore, for every term in the sum, we can think of the exponential as βQ^0 where Q^0 represents the total energy entering the amplitude $\Lambda_{\alpha_1, \cdots, \alpha_n}(k_1, \cdots, k_n)$ through the vertices with + thermal index. A generic diagram for $\Lambda_{\alpha_1, \cdots, \alpha_n}$ contains only uncircled vertices of + type, p of them, and of − type, q of them, such that $p + q = n$. Such a diagram can be represented as



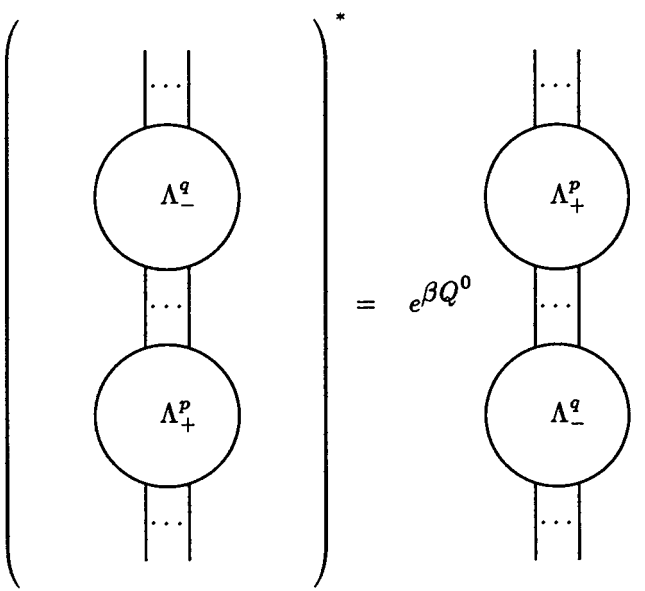
Here Λ_+^p contains only vertices (p of them) of + type and Λ_-^q contains only vertices (q of them) of – type. Let $l_1, \cdots, l_t$ represent the momenta flowing from Λ_+^p to Λ_-^q along the intermediate propagators. Let us recall some of the properties of the propagators in the momentum space (see eqs. (5.31) and (5.43)).

$$\begin{aligned} G_{++}(p) &= (G_{--}(p))^* \\ G_{+-}(p) &= (G_{+-}(p))^* = e^{-\beta p^0}\, G_{-+}(p) \end{aligned} \tag{5.50}$$

We note that if we were to complex conjugate a graph representing $\Lambda_{\alpha_1,\cdots,\alpha_n}(k_1, \cdots, k_n)$, then according to eq. (5.50),

$$\begin{aligned} G_{++}(k) &\leftrightarrow G_{--}(k) \\ G_{+-}(k) &\rightarrow (G_{+-}(k))^* = e^{-\beta k^0}\, G_{-+}(k) \end{aligned} \tag{5.51}$$

Consequently, we can think of the complex conjugate of any diagram as giving a diagram where the positive and the negative vertices are exchanged weighted by an exponential factor which adds up all the momenta flowing from the positive to the negative vertices of the original diagram and the negative of that after complex conjugation. The exponential factor, therefore, corresponds to $e^{\beta Q^0}$ where Q^0 represents the total momentum entering the complex conjugated diagram through the positive vertices. Thus, diagrammatically, we can write



(5.52)

where Q represents the total momentum entering the diagram on the right hand side of eq. (5.53) through the positive vertices. Algebraically,

this is equivalent to saying that

$$\Lambda^*_{-\alpha_1,\cdots,-\alpha_n} = e^{\beta Q^0}\ \Lambda_{\alpha_1,\cdots,\alpha_n} \tag{5.53}$$

Equation (5.53) is quite crucial for it says that we can eliminate all the exponential factors in the sum in eq. (5.49) to write

$$\begin{aligned}
\Sigma(p) &= \Gamma(p,k_1,\cdots,k_n)G_{+-}(k_1)\cdots G_{+-}(k_n) \\
&\quad \times \sum_{\alpha_1,\cdots,\alpha_n=\pm} \Lambda^*_{\alpha_1,\cdots,\alpha_n}(k_1,\cdots,k_n) \\
&= \Gamma(p,k_1,\cdots,k_n)G_{+-}(k_1)\cdots G_{+-}(k_n) \\
&\quad \times \left(\sum_{\alpha_1,\cdots,\alpha_n=\pm} \Lambda_{\alpha_1,\cdots,\alpha_n}(k_1,\cdots,k_n) \right)^*
\end{aligned} \tag{5.54}$$

The sum on the right hand side, inside the parenthesis, can now be shown to vanish through a largest time equation in the coordinate space. Each term represents a graph with only uncircled vertices of $+$ and $-$ type that are internal vertices and the thermal indices are summed over. Let x represent the coordinate of one such vertex with the largest time and we will sum over the thermal index of this vertex holding the thermal index of all other vertices fixed. This vertex will be connected to p vertices of $+$ type located at, say, $y_1,\cdots,y_p$ and q vertices of $-$ type located at $z_1,\cdots,z_q$. Then, if x^0 is the largest time coordinate, then summing over its thermal index would give rise to the factor

$$
\begin{aligned}
&\sum \Lambda \\
&\quad = R[G_{++}(x-y_1)\cdots G_{++}(x-y_p)G_{+-}(x-z_1)\cdots G_{+-}(x-z_q) \\
&\qquad - G_{-+}(x-y_1)\cdots G_{-+}(x-y_p)G_{--}(x-z_1)\cdots G_{--}(x-z_q)] \\
&\quad = R[G_{-+}(x-y_1)\cdots G_{-+}(x-y_p)G_{+-}(x-z_1)\cdots G_{+-}(x-z_q) \\
&\qquad - G_{-+}(x-y_1)\cdots G_{-+}(x-y_p)G_{+-}(x-z_1)\cdots G_{+-}(x-z_q)] \\
&\quad = 0
\end{aligned}
\tag{5.55}
$$

This, then, proves that $\Sigma(p) = 0$ (see eq. (5.54)). In other words, a diagram with an isolated island of uncircled internal vertices must vanish when summed over the internal thermal indices. This result, then, shows that diagrams of type class II as well as parts of those in class I where there occur isolated islands of internal uncircled vertices must vanish. The two results, together, then prove that when we sum over the thermal indices of all the internal vertices, the only diagrams that give nonvanishing contribution are of type class I where there are no isolated islands of internal circled/uncircled vertices.

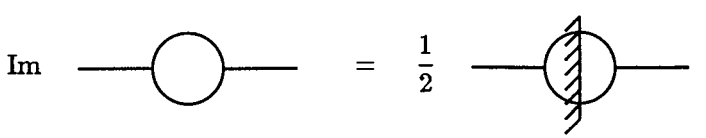
This would, therefore, seem to imply that the cutting rules generalize to finite temperature such that the imaginary part of an amplitude is given by all possible cuttings when the internal thermal indices are summed. However, there is one difference between this result and the one at zero temperature. To make things absolutely clear, let us look at the imaginary part of the self-energy which we have also discussed at zero temperature in eq. (5.28). We note that, at zero temperature, because of the properties of the cut propagators, the only diagrams that contribute to the imaginary part, by conservation of en-

ergy and momentum, are the ones where the shadow of the cut is to the left (with our convention of incoming and outgoing particles). At finite temperature, however, the cut propagators have contributions from both positive and negative energies. As a result, energy-momentum conservation allows cut diagrams to have shadows to the left as well as to the right. To make it completely parallel to the zero temperature result, we need one final ingredient. We note that if we are looking at the imaginary part of the retarded propagator, which is defined as (see eq. (2.40)),

$$G_R = G_{++} - G_{+-} = G_{-+} - G_{--} \tag{5.56}$$

then, the thermal index on the second vertex is effectively being summed (There is an extra $(-)$ factor arising from a $-$ type vertex and, hence, this really represents a sum of two amplitudes.). For such a graph, therefore, diagrams where the second (right) vertex is circled give vanishing contribution because of the first theorem that we have proved. Consequently, the only graphs that will contribute nontrivially to the imaginary part of the retarded self-energy are the ones where the circled vertices are at the left. This is quite interesting because, as we have seen in eq. (2.51), the imaginary part of the retarded Greens function is related to the Feynman Greens function through a multiplicative constant. And, what we find so far from our discussions is that, as long as we are looking at the imaginary part of the retarded function, the generalization of the cutting rules is completely parallel to what we had at zero temperature, namely, we must take all possible cuttings with the shadow on the left (with our convention of incoming and outgoing particles).

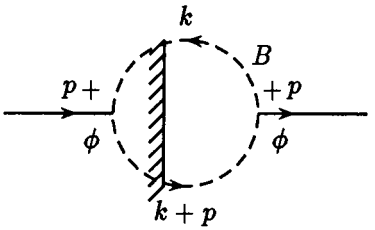
Consequently, we see that the cutting rules of zero temperature generalize to finite temperature as well – not graph by graph, but in the whole amplitude when we sum over the thermal indices of the internal vertices for a retarded amplitude. We would like to emphasize here that it is only a coincidence that at zero temperature a cutting description holds for the imaginary part, graph by graph, but unitarity only requires that such a description hold for the complete amplitude when summed over all intermediate states (channels) (see, for example, eq. (5.2)) and that is what we find at finite temperature. Diagrammatically, we can, therefore, write that at finite temperature when we sum over all the internal thermal indices, we have (e.g. for a self energy diagram), the imaginary part of the retarded self energy is given by (we suppress "retarded" in the diagram)



(5.57)

5.3 Example

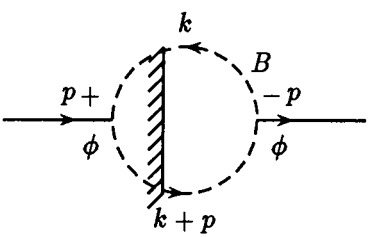
To end this discussion, we will give a simple example of the cutting rules, at one loop corresponding to the theory described in eq. (1.50). We can try to obtain the imaginary part for the retarded self-energy for the ϕ field from the cutting rules that we have described so far. We note that the cut graph with $++$ external lines has the form (remember the symmetry factor $\frac{1}{2}$)



$$
\begin{aligned}
= \quad & \frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} (2\pi)[\theta(k^0)n(|k^0|) + \theta(-k^0)(1 + n(|k^0|))]\, \delta(k^2 - M^2) \\
& \times (2\pi)[\theta(k^0 + p^0)(1 + n(|k^0 + p^0|)) + \theta(-k^0 - p^0)n(|k^0 + p^0|)] \\
& \times \delta((k + p)^2 - M^2) \\
= \quad & \frac{g^2}{8} \int \frac{d^3k}{(2\pi)^2} \frac{1}{\omega_k \omega_{k+p}} [n(\omega_k)(1 + n(\omega_{k+p}))\delta(p^0 + \omega_k - \omega_{k+p}) \\
& + (1 + n(\omega_k))n(\omega_{k+p})\delta(p^0 - \omega_k + \omega_{k+p}) \\
& + n(\omega_k)n(\omega_{k+p})\delta(p^0 + \omega_k + \omega_{k+p}) \\
& + (1 + n(\omega_k))(1 + n(\omega_{k+p}))\delta(p^0 - \omega_k - \omega_{k+p})] \qquad (5.58)
\end{aligned}
$$

where we have defined ω_k and ω_{k+p} as in eq. (1.52).

In a similar manner, we can evaluate the cut diagram with $+-$ external lines and it has the form



$$
\begin{aligned}
&= -\frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}(2\pi)[\theta(k^0)(1+n(|k^0|))+\theta(-k^0)n(|k^0|)]\,\delta(k^2-M^2) \\
&\quad \times(2\pi)[\theta(k^0+p^0)n(|k^0+p^0|\}+\theta(-k^0-p^0)(1+n(|k^0+p^0|))] \\
&\quad \times\delta((k+p)^2-M^2) \\
&= -\frac{g^2}{8}\int \frac{d^3k}{(2\pi)^2}\frac{1}{\omega_k\omega_{k+p}}[(1+n(\omega_k))n(\omega_{k+p})\delta(p^0+\omega_k-\omega_{k+p}) \\
&\quad +n(\omega_k(1+n(\omega_{k+p}))\delta(p^0-\omega_k+\omega_{k+p}) \\
&\quad +(1+n(\omega_k))(1+n(\omega_{k+p}))\delta(p^0+\omega_k+\omega_{k+p}) \\
&\quad +n(\omega_k n(\omega_{k+p})\delta(p^0-\omega_k-\omega_{k+p})] \qquad (5.59)
\end{aligned}
$$

If we add the two contributions in eqs. (5.58) and (5.59), we obtain the cut diagram for the retarded self energy and, from eq. (5.57), we see that the imaginary part of the retarded self energy is given to be

$$
\begin{aligned}
\mathrm{Im}\,\Pi_{\mathrm{R}}(p^0) = -\frac{g^2}{16}\int \frac{d^3k}{(2\pi)^2}\frac{1}{\omega_k\omega_{k+p}} \\
\times[\{(1+n(\omega_k))(1+n(\omega_{k+p}))-n(\omega_k)n(\omega_{k+p})\} \\
\times(\delta(p^0+\omega_k+\omega_{k+p})-\delta(p^0-\omega_k-\omega_{k+p})) \\
+\{n(\omega_k)(1+n(\omega_{k+p}))-n(\omega_{k+p})(1+n(\omega_k))\} \\
\times(\delta(p^0-\omega_k+\omega_{k+p})-\delta(p^0+\omega_k-\omega_{k+p}))] \qquad (5.60)
\end{aligned}
$$

This can be compared with eq. (1.64) keeping in mind the identification that $p^0=\omega$. This shows how the generalization of the cutting rules work and even though we have only worked out a simple one loop example,

it can be carried out explicitly to higher loops where more interesting structures arise.

5.4 References

Bedaque, P.F, A. Das and **S. Naik**, "Cutting Rules at Finite Temperature", hep-ph/9603325.

Cutkosky, R, J. Math. Phys. **1** (1960) 429.

't Hooft, G and **M. Veltman**, "Diagrammar", CERN 73-9.

Kobes, R and **G. Semenoff**, Nuc. Phys. **B260** (1985) 714; *ibid* **B272** (1986) 329.

Landshoff, P, Phys. Lett. **B386** (1996) 291.

Weldon, H, Phys. Rev. **D28** (1983) 2007.

Chapter 6

Symmetry Breaking

Symmetry plays a fundamental role in the study of physical phenomena. Global symmetry imposes various constraints on the structure of the theory and local symmetry which leads to physical forces is even more restrictive. Sometimes, a given symmetry in a theory may be broken and this has profound consequences. Examples of symmetry breaking that may only be too familiar are the existence of phase transitions in various physical systems where the different phases may exhibit different symmetries. We have alluded to some of the consequences of a spontaneous breakdown of a continuous global symmetry briefly in chapter 3. In such a case, the physical system develops long range correlations (in field theory language, there arise massless quanta in the theory). The consequences are even more interesting when a local symmetry is involved. We also know from the study of physical systems

such as magnets – where rotational symmetry is spontaneously broken (the magnet has a preferred direction in space) – that when the magnet is placed in a heat bath, the magnetization is lost for temperatures higher than a certain critical temperature. Above this critical temperature, the thermal motion overtakes any tendency in the elementary spins to align. As a result, the rotational symmetry which is spontaneously broken at low temperatures is restored. In this chapter, we would study such phenomena within the context of quantum field theories.

6.1 Global Symmetry

Let us, as a starting example, consider a simple scalar field theory which involves a single complex scalar field, ϕ, and is described by the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + m^2 \phi^\dagger \phi - \frac{\lambda}{2}(\phi^\dagger \phi)^2 \qquad \lambda, m^2 > 0 \qquad (6.1)$$

The complex fields can also be equivalently expressed in terms of two Hermitian fields σ and χ which is sometimes useful. Thus, let us define

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2}}(\sigma(x) + i\chi(x)) \\ \phi^\dagger(x) &= \frac{1}{\sqrt{2}}(\sigma(x) - i\chi(x)) \end{aligned} \qquad (6.2)$$

In terms of these fields, the Lagrangian density of eq. (6.1) takes the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{m^2}{2}(\sigma^2+\chi^2) - \frac{\lambda}{8}(\sigma^2+\chi^2)^2 \qquad (6.3)$$

The Lagrangian density of eq. (6.1) can be easily checked to be invariant under the infinitesimal global phase transformations (invariance under a finite transformation holds as well)

$$\begin{aligned} \delta\phi &= i\epsilon\phi \\ \delta\phi^\dagger &= -i\epsilon\phi^\dagger \end{aligned} \qquad (6.4)$$

where ϵ is a constant, infinitesimal, real parameter of the phase transformation. Equivalently, we can show that the Lagrangian density of eq. (6.3) is invariant under the infinitesimal global transformations

$$\begin{aligned} \delta\sigma &= -\epsilon\chi \\ \delta\chi &= \epsilon\sigma \end{aligned} \qquad (6.5)$$

The Nöether current density associated with the symmetry transformations, eq. (6.4) or (6.5), can be derived in a straightforward manner and has the form

$$\begin{aligned} J^\mu &= i(\phi^\dagger\partial^\mu\phi - \partial^\mu\phi^\dagger\phi) \\ &= (\chi\partial^\mu\sigma - \sigma\partial^\mu\chi) \end{aligned} \qquad (6.6)$$

from which the conserved charge is obtained to be

$$\begin{aligned} Q = \int d^3x\, J^0 &= \int d^3x\, i(\phi^\dagger\dot{\phi} - \dot{\phi}^\dagger\phi) \\ &= \int d^3x\, (\chi\dot{\sigma} - \sigma\dot{\chi}) \end{aligned} \qquad (6.7)$$

The charge, Q, is the generator of the infinitesimal transformations in eq. (6.4) or (6.5) and since these define a symmetry of the theory (leave the Hamiltonian invariant), we can check (with the canonical commutation relations of the theory) that Q commutes with the Hamiltonian, namely,

$$[Q, H] = 0 \tag{6.8}$$

with H representing the Hamiltonian which governs the dynamics of the theory and has the form (we are going to be slightly sloppy – the $\dot{\phi}$ and $\dot{\phi}^\dagger$ etc should really be replaced by the appropriate conjugate momenta)

$$\begin{aligned} H &= \int d^3x \left[\dot{\phi}^\dagger\dot{\phi} + \vec{\nabla}\phi^\dagger \cdot \vec{\nabla}\phi - m^2\phi^\dagger\phi + \frac{\lambda}{2}(\phi^\dagger\phi)^2\right] \\ &= \int d^3x \left[\frac{1}{2}(\dot{\sigma}^2 + \dot{\chi}^2) + \frac{1}{2}(\vec{\nabla}\sigma \cdot \vec{\nabla}\sigma + \vec{\nabla}\chi \cdot \vec{\nabla}\chi)\right. \\ &\qquad \left. - \frac{m^2}{2}(\sigma^2 + \chi^2) + \frac{\lambda}{8}(\sigma^2 + \chi^2)^2\right] \end{aligned} \tag{6.9}$$

Furthermore, being the generator of the symmetry transformations, the commutator of Q with the field variables gives the infinitesimal transformations (with a factor of i and without the parameter of transformation ϵ), namely,

$$\begin{aligned} [Q, \phi(x)] &= i\delta\phi(x) \\ \left[Q, \phi^\dagger(x)\right] &= i\delta\phi^\dagger(x) \\ [Q, \sigma(x)] &= i\delta\sigma(x) \\ [Q, \chi(x)] &= i\delta\chi(x) \end{aligned} \tag{6.10}$$

While the Lagrangian density (or the Hamiltonian) of the theory is invariant under the transformations in eq. (6.4) or (6.5), the ground state of the theory, in this case, is not. This is more easily seen in terms of the fields σ and χ as follows. First, we note that the theory in eq. (6.1) or (6.3) is different from the standard theory for a complex scalar field in that the sign of the mass term is reversed. Second, we note that the equation of motion following from the Lagrangian density in eq. (6.3) (the Euler-Lagrange equations) are given by

$$\begin{aligned} \left(\Box - m^2 + \frac{\lambda}{2}(\sigma^2+\chi^2)\right)\sigma &= 0 \\ \left(\Box - m^2 + \frac{\lambda}{2}(\sigma^2+\chi^2)\right)\chi &= 0 \end{aligned} \tag{6.11}$$

It is clear from the form of H that constant field configuration ($\sigma(x) = \sigma = \text{constant}$; $\chi(x) = \chi = \text{constant}$) would lead to a minimum of energy (the kinetic energy terms in this case will be zero) and also would be a solution of the equations of motion, eq. (6.11), provided

$$\frac{\partial V}{\partial \sigma} = 0 = \frac{\partial V}{\partial \chi} \tag{6.12}$$

where we note, from eq. (6.9), that

$$V(\sigma,\chi) = -\frac{m^2}{2}(\sigma^2+\chi^2) + \frac{\lambda}{8}(\sigma^2+\chi^2)^2 \tag{6.13}$$

which is known as the tree level potential. For constant σ, χ, it is a simple function and that is the reason why we do not have functional derivatives in eq. (6.12).

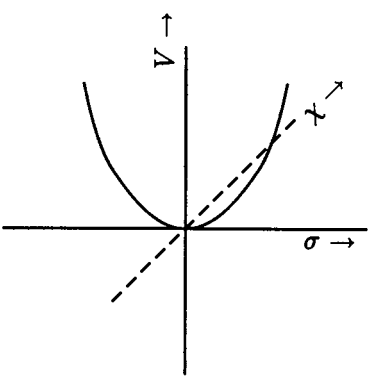
From eqs. (6.12) and (6.13), we see that, for $\lambda, m^2 > 0$, the extremum of the potential occurs where

$$\sigma = 0 = \chi$$

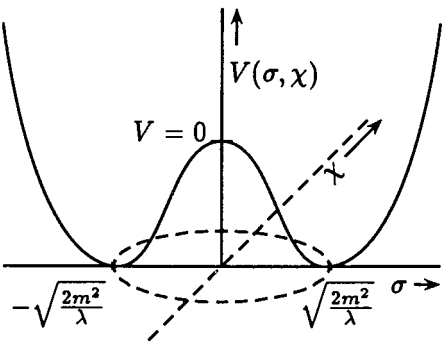
$$\text{or,} \quad (\sigma^2 + \chi^2) = \frac{2m^2}{\lambda} \tag{6.14}$$

It can be easily checked that the first solution representing the origin in the field configuration space is a local maximum whereas it is the second solution which represents the true minimum. This is the difference between this theory and the standard complex scalar field theory where the mass term has a negative sign (in the Lagrangian density) which leads to the minimum of the potential at the origin of the field configuration space. It is also clear that the second solution giving the minimum, in eq. (6.14), defines a circle in the field space which is invariant under the transformations in eq. (6.5). Every point on this circle defines a minimum of energy and, consequently, we have an infinitely degenerate set of minima. However, once we choose a particular point on this circle as representing the minimum of the potential of our theory, the global invariance of eq. (6.5) would be broken.

$\underline{m^2 < 0}$:



$\underline{m^2 > 0}$:



Let us, for simplicity, choose

$$\sigma = v = \sqrt{\frac{2m^2}{\lambda}} \qquad \chi = 0 \tag{6.15}$$

as the minimum of the potential of our theory. In the quantum theory, as we will see later, this corresponds to choosing a ground state such that

$$\langle 0|\sigma|0\rangle = \langle\sigma\rangle = v \qquad \langle 0|\chi|0\rangle = \langle\chi\rangle = 0 \tag{6.16}$$

Consequently, $v = \sqrt{\frac{2m^2}{\lambda}}$ is also known as the vacuum expectation value of the σ field. It is now easy to see that the global symmetry of eq. (6.5) is spontaneously broken as follows. From eqs. (6.5), (6.10) and (6.16), we get

$$\langle [Q,\chi]\rangle = i\delta\langle\chi\rangle = i\langle\sigma\rangle \neq 0 \tag{6.17}$$

For this to be true, the generator of infinitesimal transformations, Q, cannot annhilate the ground state, namely, (otherwise, the left hand side of eq. (6.17) would vanish)

$$Q|0\rangle \neq 0 \tag{6.18}$$

Equivalently, the ground state or the vacuum will not be invariant under the group of global transformations. (More intuitively, every point on the circle of minima in eq. (6.14) defines a vacuum of the quantized theory. Under the action of the global transformations, each point on this circle maps to another point and correspondingly, a given vacuum of the quantized theory under the action of the group of transformations goes to an inequivalent ground state.)

The field whose transformation picks up a nonzero vacuum expectation value can be shown to become massless and is known as the Goldstone boson. Thus, in the present case, we see from eq. (6.17), that χ would represent the Goldstone boson field. To see that this field, indeed, becomes massless, we note that we can define our quantum theory by redefining the field variables as

$$\begin{aligned} \sigma &\rightarrow \sigma + v \\ \chi &\rightarrow \chi \end{aligned} \tag{6.19}$$

so that the new fields would have zero vacuum expectation value and this is what we would expect for a normal field theory. With this redefinition, the Lagrangian density of eq. (6.3) becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - m^2\sigma^2 + \frac{m^4}{2\lambda} \\ &\quad -m\sqrt{\frac{\lambda}{2}}\sigma(\sigma^2+\chi^2) - \frac{\lambda}{8}(\sigma^2+\chi^2)^2 \end{aligned} \tag{6.20}$$

This clearly shows that the σ-field has become massive with a mass $\sqrt{2}m$ while the χ-field has become massless. This is the Goldstone

boson. According to Goldstone's theorem, whenever a continuous symmetry is spontaneously broken in a manifestly Lorentz invariant theory with a positive metric Hilbert space, massless particles must arise as a consequence of symmetry breaking. Here, we have explicitly seen this to be the case. More intuitively, Goldstone's theorem can be understood from the structure of the potential of the theory as follows. We have seen that the potential, in the case of spontaneous symmetry breaking has the structure of a double well, rotated around the vertical axis. (This is also known as a Mexican hat potential.) The true minimum of the potential, which occurs for a finite value of the field variables, allows for two kinds of motion. The small oscillations in the well, obviously, are harmonic and would correspond to massive quanta. On the other hand, the motion along the valley of the minimum would cost no energy and, therefore, would correspond to massless quanta. That is physically the origin of Goldstone bosons (massless particles). On the other hand, the existence of massless quanta in the theory would imply long range correlations in the theory. As we will see in the next section, in the presence of gauge interactions, however, such massless particles are naturally removed.

6.2 Local Symmetry

Let us consider, once again, the complex scalar field theory of the previous section where the complex fields now interact with an Abelian gauge field. (This is only for simplicity of discussions.) The Lagrangian density now has the form (see the discussion in chapter 4, namely, 4.1)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi + m^2\phi^\dagger\phi - \frac{\lambda}{2}(\phi^\dagger\phi)^2 \tag{6.21}$$

Here, the covariant derivative is defined as (see eq. (4.10))

$$D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi \tag{6.22}$$

where A_μ represents the Abelian gauge field and e the coupling constant or the interaction strength. This model is commonly known as the Abelian Higgs model.

The Lagrangian density, (6.21), can also be written in the σ-χ basis and takes the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{m^2}{2}(\sigma^2+\chi^2)$$
$$+e(\chi\partial^\mu\sigma - \sigma\partial^\mu\chi)A_\mu + \frac{e^2}{2}(\sigma^2+\chi^2)A_\mu A^\mu - \frac{\lambda}{8}(\sigma^2+\chi^2)^2 \tag{6.23}$$

The Lagrangian densities in eq. (6.21) or (6.23) can be easily seen to be invariant under the transformations in eq. (6.4) or (6.5) respectively with the parameter of transformation a local function of space-time with the gauge fields also transforming as in eq. (4.13) (note that $f^{abc} = 0$ for an Abelian theory), namely,

$$\begin{aligned} \delta\phi &= ie\epsilon(x)\phi \\ \delta\phi^\dagger &= -ie\epsilon(x)\phi^\dagger \\ \delta A_\mu &= \partial_\mu\epsilon \end{aligned} \tag{6.24}$$

Or, equivalently, we can write them in terms of the σ and the χ fields as

$$\begin{aligned} \delta\sigma &= -e\epsilon(x)\chi \\ \delta\chi &= e\epsilon(x)\sigma \\ \delta A_\mu &= \partial_\mu\epsilon(x) \end{aligned} \tag{6.25}$$

The global invariance of the theory, therefore, has been promoted to a local invariance because of the presence of the Abelian gauge field.

The scalar potential of the theory, however, is unchanged from the discussions of the previous section. Consequently, following our discussion of the last section, we note that (see eq. (6.15)) the minimum of the potential occurs at nontrivial values of the field configuration

$$\langle\sigma\rangle = v \qquad \langle\chi\rangle = 0$$

The continuous phase symmetry is spontaneously broken and, again, we can expand our quantum field theory around this nontrivial vacuum (see eq. (6.19)) which has the form

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - m^2\sigma^2 + \frac{m^4}{2\lambda} \\ &+ \frac{e^2v^2}{2}A_\mu A^\mu - ev\partial_\mu\chi A^\mu + e(\chi\partial_\mu\sigma - \sigma\partial_\mu\chi)A^\mu + e^2v\sigma A_\mu A^\mu \\ &+ \frac{e^2}{2}(\sigma^2+\chi^2)A_\mu A^\mu - \frac{\lambda v}{2}\sigma(\sigma^2+\chi^2) - \frac{\lambda}{8}(\sigma^2+\chi^2)^2 \end{aligned} \tag{6.26}$$

Equation (6.26) is quite interesting in that it shows that the Abelian gauge field and the "Goldstone" field, χ, mix because of the symmetry breaking. To see the spectrum of the theory, we need

to consider only the quadratic part of the Lagrangian density in (6.26). This has the form

$$\begin{aligned}\mathcal{L}_{Qua} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - m^2\sigma^2 \\ &\quad -ev\partial_\mu\chi A^\mu + \frac{e^2v^2}{2}A_\mu A^\mu \end{aligned} \tag{6.27}$$

This has the interesting structure that if we define a new gauge field as

$$B_\mu = A_\mu - \frac{1}{ev}\,\partial_\mu\chi \tag{6.28}$$

then, the quadratic Lagrangian density of eq. (6.27) can be written as

$$\begin{aligned}\mathcal{L}_{Qua} &= -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)(\partial^\mu B^\nu - \partial^\nu B^\mu) + \frac{e^2v^2}{2}B_\mu B^\mu \\ &\quad +\frac{1}{2}\,\partial_\mu\sigma\partial^\mu\sigma - m^2\sigma^2 \end{aligned} \tag{6.29}$$

We see that with the redefinition in eq. (6.28), the "Goldstone" field (massless χ field) has disappeared completely from the theory. The redefinition in eq. (6.28), of course, has precisely the same form as a gauge transformation (see eq. (6.25)) and so it is clear that the χ field has been absorbed into the gauge degrees of freedom of the A_μ field. Furthermore, we also see that the Abelian gauge field (photon) has now become massive with a mass

$$m_{gauge} = ev = e\sqrt{\frac{2m^2}{\lambda}} \tag{6.30}$$

Thus, we see that, in this case, we have a massive gauge field with mass m_{gauge} and a scalar field (σ) with mass $\sqrt{2}m$, but there is no massless scalar field any more. The χ degree of freedom has gone into

the massive degree of freedom of the gauge field and so the number of physical degrees of freedom is maintained (there is no violation of unitarity). Conventionally, one says that the gauge field has eaten up the "Goldstone" field to become massive.

The "Goldstone" field has, of course, disappeared from the quadratic part of the Lagrangian density, but we can also show that it does not have any physical effect in the full theory. This can be seen most easily by noting that we have a gauge theory with a local gauge invariance. As we have discussed in chapter 4, in such a case, we need to choose a gauge for any physical, perturbative calculation. (We point out here that in a lattice calculation, in contrast, there is no need for a choice of gauge.) In particular, we can choose a gauge where

$$\chi(x) = 0 \tag{6.31}$$

With this choice of gauge, the Lagrangian density of eq. (6.26) including interactions would become

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - m^2\sigma^2 + \frac{e^2v^2}{2}A_\mu A^\mu + \frac{m^4}{2\lambda} \\ &+ e^2v\sigma A_\mu A^\mu + \frac{e^2}{2}\sigma^2 A_\mu A^\mu - \frac{\lambda v}{2}\sigma^3 - \frac{\lambda}{8}\sigma^4 \end{aligned} \tag{6.32}$$

In this gauge, the Lagrangian density is written completely in terms of physical field variables and, therefore, such a gauge choice is also known as the unitary gauge. However, this is not a particularly convenient gauge choice from the point of view of calculations. A popular gauge which is convenient for calculations is known as the R_ξ-gauge. Here one adds to the Lagrangian density of eq. (6.26) a gauge fixing and ghost

Lagrangian density of the form (without the auxiliary fields of chapter 4, see, for example, eq. (4.23) and (6.25))

$$\begin{aligned}
&\mathcal{L}_{gf} + \mathcal{L}_{ghost} \\
&\quad = -\frac{1}{2\xi}(\partial_\mu A^\mu + \xi ev\chi)^2 + \partial_\mu \bar{c}\partial^\mu c - \xi(ev)^2 \bar{c}c - \xi e^2 v\sigma \bar{c}c \\
&\quad = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 - ev\chi\partial_\mu A^\mu - \frac{\xi(ev)^2}{2}\chi^2 + \partial_\mu \bar{c}\partial^\mu c \\
&\qquad -\xi(ev)^2 \bar{c}c - \xi e^2 v\sigma \bar{c}c
\end{aligned} \tag{6.33}$$

Comparing this with eq. (6.26), we see that the mixing between χ and the A_μ fields would cancel in this gauge. However, both the χ field and the ghost fields would pick up an identical, gauge dependent mass (see also eq. (6.30))

$$m_\chi^2 = m_{ghost}^2 = \xi\,(ev)^2 = \xi\, m_{gauge}^2 \tag{6.34}$$

We would end this section with a brief discussion of how the "Goldstone" theorem fails in this case. We recall that the "Goldstone" theorem necessarily predicts the presence of massless particles in a theory when a continuous symmetry is spontaneously broken. The assumption, in the theorem, however, is that the theory is manifestly Lorentz invariant with a positive metric for the Hilbert space. These assumptions are clearly violated in the case of a gauge theory. As we have seen in chapter 4, if we want to describe a gauge theory in a manifestly Lorentz invariant manner, we must necessarily have unphysical particles with negative norm and consequently, we cannot have a positive metric for the Hilbert space. Conversely, if we choose to describe

the theory only with the physical degrees of freedom – the transverse components –then, the Hilbert space will have a positive definite inner product, but manifest Lorentz invariance will be lost. Thus, we see that in the presence of a local invariance, the basic assumptions of the "Goldstone" theorem fail and, consequently, we avoid the presence of massless fields (particles).

6.3 Effective Potential

It is clear from the discussion of the last two sections that the classical potential of a quantum field theory plays a crucial role in studying the question of symmetry breaking. However, sometimes the radiative corrections in a quantum theory may change the behavior of the classical potential. The minimum of the classical potential may be unstable under radiative corrections. A symmetry that is spontaneously broken at the classical level may be restored or even a symmetry that is unbroken at the classical level may be spontaneously broken because of quantum effects. The important quantity to consider, for this reason, is the effective potential which takes into account the quantum corrections in a theory. The effective potential, unfortunately, does not have a closed form expression unlike the classical potential and, for this reason, one has to analyze this quantity order by order. In our discussions, we will restrict to the effective potential taking into account only the one loop corrections. There are, however, systematic procedures to go beyond one loop.

Let us, for simplicity, consider a scalar ϕ^4-theory in the presence of a source described by the action (The source allows us to define Greens functions of the theory.)

$$S^J = \int d^4x[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi] = S + \int d^4x J\phi \quad (6.35)$$

The classical or the tree level potential clearly is given by (with $J = 0$)

$$V^0 = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \quad (6.36)$$

The generating functional, in this case, is defined to be

$$Z[J] = e^{\frac{i}{\hbar}W[J]} = \int \mathcal{D}\phi\, e^{\frac{i}{\hbar}S^J} \quad (6.37)$$

Here, $W[J]$ is known as the generating functional for the connected Greens functions which gives

$$\frac{\delta^n W}{\delta J(x_1)\cdots\delta J(x_n)}\Big|_{J=0} = (i\hbar)^{n-1}\langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle^c \quad (6.38)$$

From equation (6.38), we note, in particular, that the one point function is given by

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} = \langle 0|\phi(x)|0\rangle_J \quad (6.39)$$

From the above definition, we see that the classical field, ϕ_c, which represents the vacuum expectation value of the field, ϕ, would be a constant independent of space-time when the source is turned off simply because of the Lorentz invariance of the vacuum. In general, however,

it will be a functional of the source J and is known to generate the tree diagrams of the theory.

The Euler-Lagrange equation following from the form of the action in eq. (6.35) is given by

$$\begin{aligned} \frac{\delta S^J}{\delta\phi(x)} = \frac{\delta S}{\delta\phi(x)} + J(x) &= 0 \\ \text{or,}\quad (\Box + m^2)\phi + \frac{\lambda}{3!}\phi^3 - J &= 0 \end{aligned} \tag{6.40}$$

For simplicity, let us denote this as

$$F(\phi) = J \tag{6.41}$$

where $F(\phi)$ represents the Euler operator. Next, we note from the definition of the generating functional in eq. (6.37) that it has no dependence on the field variable. (All possible field configurations are being integrated over in the path integral.) Consequently, if we were to redefine

$$\phi \to \phi + \delta\phi$$

with $\delta\phi$ arbitrary under the integral, the generating functional would not change. This, therefore, leads to the identity (under such a field redefinition)

$$\delta Z[J] = 0 \tag{6.42}$$

This, in turn, would imply that (see eqs. (6.37) and (6.40))

$$\begin{aligned}\int \mathcal{D}\phi \frac{\delta S^J}{\delta \phi}\, e^{iS^J} &= 0 \\ &= \int \mathcal{D}\phi\, (F(\phi) - J)\, e^{iS^J} \\ &= \int \mathcal{D}\phi\, (F(\frac{\hbar}{i}\frac{\delta}{\delta J}) - J)\, e^{iS^J} \\ &= (F(\frac{\hbar}{i}\frac{\delta}{\delta J}) - J)Z[J]\end{aligned}$$

In terms of $W[J]$, this, then, gives

$$\begin{aligned}e^{-\frac{i}{\hbar}W[J]} F(\frac{\hbar}{i}\frac{\delta}{\delta J(x)})\, e^{\frac{i}{\hbar}W[J]} &= J(x) \\ \text{or, } F(\frac{\delta W}{\delta J(x)} + \frac{\hbar}{i}\frac{\delta}{\delta J(x)}) &= J(x) \qquad (6.43)\end{aligned}$$

(Here, we are assuming that the functional derivative, after simplification, acts on the identity on the right.) In the limit of vanishing $\hbar$, this equation reduces to (see eq. (6.39))

$$\begin{aligned}F(\frac{\delta W}{\delta J(x)}) &= J(x) \\ \text{or, } F(\phi_c) &= J(x) \qquad (6.44)\end{aligned}$$

Thus, we see that the classical field satisfies the same equation as the Euler-Lagrange equation of the original theory (see eq. (6.41)) when $\hbar \to 0$. If we now switch off the source, then, as we have noted earlier, the classical field would only have a constant value. Consequently, in that limit, the equation that would result from eq. (6.44) is

$$\frac{\partial V^0(\phi_c)}{\partial \phi_c} = 0 \tag{6.45}$$

This is the minimum equation for the potential in terms of ϕ_c and as we have seen earlier, a nontrivial minimum of the tree level potential would correspond to a nontrivial solution for eq. (6.45). On the other hand, a nontrivial solution for (6.45) would imply

$$\phi_c = \langle 0|\phi|0\rangle \neq 0 \tag{6.46}$$

This, therefore, ties in with our earlier assertion (see eqs. (6.15), (6.16) and the discussion there) that a nontrivial minimum for the classical potential would imply that the quantum fields would have nonzero vacuum expectation value in the theory. We can think of $\phi_c = \langle 0|\phi|0\rangle$ as an order parameter of symmetry breaking and functionally, we would recognize spontaneous symmetry breaking if (see eqs. (6.39) and (6.46))

$$\left.\frac{\delta W}{\delta J}\right|_{J=0} \neq 0 \tag{6.47}$$

In a quantum field theory, however, we are more interested in studying the effective action or the generating functional for the 1PI (one particle irreducible) graphs which is the Legendre transformation of $W[J]$ defined as follows

$$\Gamma[\phi_c] = W[J] - \int d^4x\, J(x)\phi_c(x) \tag{6.48}$$

This definition immediately leads to

$$\frac{\delta \Gamma}{\delta \phi_c(x)} = -J(x) \tag{6.49}$$

The symmetry breaking condition of eq. (6.47) can now be written as

$$\left.\frac{\delta\Gamma}{\delta\phi_c}\right|_{\phi_c=v\neq 0} = 0 \tag{6.50}$$

As we will see shortly, this condition is more like the minimum condition of eq. (6.45).

The effective action, Γ, is a functional of the classical field and there are two equivalent ways of expanding it. First, we can expand it as a power series in ϕ_c to write

$$\Gamma[\phi_c] = \sum_n \int d^4x_1 \cdots d^4x_n \, \phi_c(x_1)\cdots\phi_c(x_n)\,\Gamma^n(x_1,\cdots,x_n) \tag{6.51}$$

where the coefficients of expansion

$$\Gamma^n(x_1,\cdots,x_n) = \left.\frac{\delta^n\Gamma}{\delta\phi_c(x_1)\cdots\delta\phi_c(x_n)}\right|_{\phi_c=0} \tag{6.52}$$

represent the 1PI vertex functions. This expansion assumes that there is no symmetry breaking. On the other hand, if we had a spontaneous breakdown of symmetry, the corresponding formulae would take the form

$$\Gamma[\phi_c] = \sum_n \int d^4x_1 \cdots d^4x_n \,(\phi_c(x_1)-v)\cdots(\phi_c(x_n)-v)\,\Gamma^n \tag{6.53}$$

with the 1PI vertex functions now defined as

$$\Gamma^n(x_1,\cdots,x_n) = \left.\frac{\delta^n\Gamma}{\delta\phi_c(x_1)\cdots\delta\phi_c(x_n)}\right|_{\phi_c=v} \tag{6.54}$$

with v denoting the solution of the minimum equation (6.50).

An alternate expansion of the effective action is in terms of powers of momentum (or derivatives) as

$$\Gamma[\phi_c] = \int d^4x \, [-V(\phi_c) + \frac{1}{2}\partial_\mu\phi_c\partial^\mu\phi_c + \cdots] \tag{6.55}$$

Here the "$\cdots$" represent terms which involve higher derivatives of ϕ_c. It is easy to see that such an expansion is possible simply because Γ represents an effective action and must necessarily contain the structure of the classical action in eq. (6.35). This form of the expansion of the effective action, however, is more convenient from the point of view of the study of symmetry breaking. Thus, for example, we have seen that when $J \to 0$, $\phi_c(x) = \phi$ is a constant (which may or may not be zero). In this limit, then, all the derivative terms in the expansion in eq. (6.55) vanish and we have

$$\Gamma[\phi] = -(2\pi)^4\delta(0)\, V(\phi) = -\Omega\, V(\phi) \tag{6.56}$$

where $V(\phi)$ is known as the effective potential. The multiplicative factor, Ω, simply represents the space-time volume and because ϕ is a constant, we only have a function (and not a functional) of this variable. Consequently, the minimum equation of (6.50) becomes

$$\frac{\partial V(\phi)}{\partial\phi} = 0 \tag{6.57}$$

which is more like eq. (6.45). However, V now contains quantum corrections to the original classical potential of eq. (6.36).

6.4 One Loop Potential

As we have noted earlier, V represents the effective potential including quantum corrections of all orders. However, unlike the classical (tree level) potential, it does not have a closed form. Consequently, its structure has to be studied order by order in perturbation theory. A perturbative expansion, which is particularly convenient in connection with symmetry breaking, is in terms of the number of loops of a Feynman diagram which also coincides with an expansion in powers of $\hbar$. In our discussions, we will restrict ourselves to one loop for simplicity, although the calculations can be carried out systematically to higher loops as well. Thus, we write

$$V = V^0 + \hbar V^1 + O(\hbar^2) \tag{6.58}$$

where we know the tree level potential, V^0, from eq. (6.36), and we are interested in calculating the one loop corrections, V^1.

There are several ways of calculating the one loop corrections to the effective potential and we will use the one that naturally generalizes to the case of finite temperature as well. Let us assume that ϕ_0 represents a solution of the tree level Euler-Lagrange equation

$$\left.\frac{\delta S^J(\phi)}{\delta\phi(x)}\right|_{\phi=\phi_0} = 0 \tag{6.59}$$

In such a case, we can expand the field variables around this classical solution as

$$\phi(x) = \phi_0(x) + \sqrt{\hbar}\chi(x) \tag{6.60}$$

Correspondingly, the classical action would have the expansion

$$\begin{aligned} S^J(\phi) &= S^J(\phi_0 + \sqrt{\hbar}\chi) \\ &= S^J(\phi_0) + \sqrt{\hbar}\int d^4x\, \chi(x)\, \frac{\delta S^J}{\delta\phi(x)}\bigg|_{\phi=\phi_0} \\ &\quad + \frac{\hbar}{2}\int d^4x d^4y\, \chi(x)\, \frac{\delta^2 S^J}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=\phi_0} \chi(y) + \cdots \\ &= S^J(\phi_0) - \frac{\hbar}{2}\int d^4x d^4y\, \chi(x) G^{-1}(x,y)\chi(y) + \cdots \end{aligned} \tag{6.61}$$

There are several things to note about the expansion in eq. (6.61). First, the linear term in χ vanishes because of eq. (6.59), namely, because ϕ_0 represents a solution of the classical Euler-Lagrange equation. Second, the terms represented by "..." are higher order in the fields χ and, therefore, from the expansion in eq. (6.60), we see that they would contribute with a higher power of $\hbar$. Consequently, we can neglect them. Furthermore, since the source terms in S^J are linear in the field variables, they do not contribute to the second derivative terms and we have defined

$$G^{-1}(x,y) = -\frac{\delta^2 S^J}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi_0} = -\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi_0} \tag{6.62}$$

This is simply an inverse propagator in the presence of a background field and, as we will see shortly, would correspond to a propagator with an effective mass in the calculation of the effective potential.

Substituting the expansion in eq. (6.61) (keeping only up to quadratic terms in χ) into the expression for the generating functional in eq. (6.37), we have

$$\begin{aligned} Z[J] &= e^{\frac{i}{\hbar}W[J]} = \int \mathcal{D}\chi \, e^{\frac{i}{\hbar}S^J(\phi_0+\sqrt{\hbar}\chi)} \\ &= e^{\frac{i}{\hbar}S^J(\phi_0)} \int \mathcal{D}\chi \, e^{-\frac{i}{2}\int d^4x d^4y \, \chi(x)G^{-1}(x,y)\chi(y)} \\ &= e^{\frac{i}{\hbar}S^J(\phi_0)} \, [\det(G^{-1})]^{-1/2} \end{aligned} \tag{6.63}$$

It follows, therefore, that

$$\begin{aligned} W[J] &= S^J(\phi_0) + \frac{\hbar}{i} \ln [\det(G^{-1})]^{-1/2} \\ &= S^J(\phi_0) + \frac{i\hbar}{2} \operatorname{Tr} \ln G^{-1} \end{aligned} \tag{6.64}$$

The determinant in the above equation, of course, needs to be evaluated carefully by properly rotating to the Euclidean space, but we will come to that shortly.

To go to the effective action, we note that ϕ_c, the classical field, can also be expanded in powers of $\hbar$. However, since it satisfies the classical equation in the limit $\hbar \to 0$ (see eq. (6.44)), we can write its expansion as

$$\phi_c(x) = \phi_0(x) + \phi_1(x) + \cdots \tag{6.65}$$

where by definition, ϕ_1 is of order $\hbar$ while the "$\cdots$" represent higher order terms in the expansion. With this, we note now that

$$\begin{aligned} S(\phi_c) &\sim S(\phi_0+\phi_1) \\ &= S(\phi_0)+\int d^4x\phi_1 \left.\frac{\delta S}{\delta\phi(x)}\right|_{\phi_0} + O(\hbar^2) \\ &= S(\phi_0)-\int d^4x J(x)\phi_1(x) \end{aligned} \tag{6.66}$$

where we have used the classical Euler lagrange equation (6.40) and have neglected the higher order terms.

It now follows from eqs. (6.48) and (6.64) that the effective action, to this order, has the form

$$\begin{aligned} \Gamma[\phi_c] &= W[J]-\int d^4x\,\phi_c(x)J(x) \\ &= S^J(\phi_0)+\frac{i\hbar}{2}\,\mathrm{Tr}\ln G^{-1}-\int d^4x\,(\phi_0+\phi_1)J \\ &= S(\phi_0)+\int d^4x\,\phi_0 J+\frac{i\hbar}{2}\,\mathrm{Tr}\ln G^{-1}-\int d^4x\,(\phi_0+\phi_1)J \\ &= S(\phi_0)-\int d^4x\,\phi_1 J+\frac{i\hbar}{2}\,\mathrm{Tr}\ln G^{-1} \\ &= S(\phi_c)+\frac{i\hbar}{2}\,\mathrm{Tr}\ln G^{-1} \end{aligned} \tag{6.67}$$

Here we used the identification in eq. (6.66) in the last step. For constant field configurations, then, we obtain the effective potential up to this order as

$$\begin{aligned} \Gamma[\phi_c]=-\Omega\,V(\phi_c) &= -\Omega\,V^0(\phi_c)+\frac{i\hbar}{2}\,\mathrm{Tr}\ln G^{-1} \\ \text{or,}\quad V(\phi_c) &= V^0(\phi_c)-\frac{i\hbar}{2}\,\Omega^{-1}\,\mathrm{Tr}\ln G^{-1} \end{aligned} \tag{6.68}$$

With the identification in eq. (6.58), we now see that the one loop contribution to the effective potential is given by

$$V^1(\phi_c) = -\frac{i\hbar}{2}\,\Omega^{-1}\,\mathrm{Tr}\ln G^{-1} \tag{6.69}$$

To explicitly evaluate the one loop correction to the potential, we first note, from the definition in eq. (6.62) and the form of S in eq. (6.35), that

$$\begin{aligned}\langle x|G^{-1}|y\rangle = G^{-1}(x,y) &= (\Box + m^2 + \frac{\lambda}{2}\phi_0^2)\delta^4(x-y) \\ &= (\Box + m^2 + \frac{\lambda}{2}\phi_c^2)\delta^4(x-y) \\ &= (\Box + m_{eff}^2)\delta^4(x-y) \end{aligned} \tag{6.70}$$

where we have used the fact that to the order of our calculation, we can use $\phi_c \sim \phi_0$ (remember that the expression in eq. (6.69) has an overall factor of $\hbar$). This also shows, as we had noted earlier, that for the calculation of the effective potential, G behaves like a propagator with an effective mass

$$m_{eff}^2 = m^2 + \frac{\lambda}{2}\phi_c^2 \tag{6.71}$$

Even though G^{-1} is not diagonal in the coordinate basis, it is diagonal in the momentum basis and has the form

$$\langle k|G^{-1}|k'\rangle = G^{-1}(k,k') = (-k^2 + m_{eff}^2)\delta^4(k-k') \tag{6.72}$$

Consequently, using the fact that for a diagonal matrix we have,

$$\langle i|\ln A_D|j\rangle = \ln\langle i|A_D|j\rangle \tag{6.73}$$

we can now obtain

$$\begin{aligned}
\mathrm{Tr}\ln G^{-1} &= \int d^4x\ \langle x|\ln G^{-1}|x\rangle \\
&= \int d^4x d^4k d^4k'\ \langle x|k\rangle\langle k|\ln G^{-1}|k'\rangle\langle k'|x\rangle \\
&= \int d^4x d^4k d^4k'\ \frac{e^{-i(k-k')\cdot x}}{(2\pi)^4}\ \ln(-k^2+m_{eff}^2)\ \delta^4(k-k') \\
&= \int d^4x \frac{d^4k}{(2\pi)^4}\ \ln(-k^2+m_{eff}^2) \\
&= \Omega \int \frac{d^4k}{(2\pi)^4}\ \ln(-k^2+m_{eff}^2)
\end{aligned} \tag{6.74}$$

Thus, from eq. (6.69), we see that we can identify

$$V^1(\phi_c) = -\frac{i\hbar}{2}\int \frac{d^4k}{(2\pi)^4}\ \ln(-k^2+m_{eff}^2) \tag{6.75}$$

This integral can be evaluated by rotating to the Euclidean space with

$$k^0 \to ik_4$$

so that the one loop correction becomes

$$\begin{aligned}
V^1(\phi_c) &= -\frac{i\hbar}{2}\int \frac{id^4k_E}{(2\pi)^4}\ \ln(k_E^2+m_{eff}^2) \\
&= \frac{\hbar}{2}\frac{2\pi^2}{(2\pi)^4}\int_0^\infty dk_E\ k_E^3\ \ln(k_E^2+m_{eff}^2)
\end{aligned} \tag{6.76}$$

This integral diverges at the upper limit and hence needs to be regulated. Using a large cut-off, Λ, we can evaluate the integral to be

$$\begin{aligned} V^1(\phi_c) &= \frac{\hbar}{32\pi^2}\int_0^{\Lambda^2} dx\; x\; \ln(x+m_{eff}^2) \\ &= \frac{\hbar}{32\pi^2}\,[\,\frac{\Lambda^4}{2}(\ln\Lambda^2-\frac{1}{2})+m_{eff}^2\Lambda^2 \\ &\quad +\frac{m_{eff}^4}{2}(\ln\frac{m_{eff}^2}{\Lambda^2}-\frac{1}{2})+O(\frac{1}{\Lambda^2})\,] \end{aligned} \tag{6.77}$$

The important point to note is that the one loop correction is explicitly divergent as $\Lambda \to \infty$. Consequently, we must add to our theory counterterms which would take care of these divergences. This is the standard renormalization procedure in a quantum field theory. If we add counterterms and renormalize our theory by demanding that the renormalized mass and the coupling constants of the theory up to one loop are given by

$$\begin{aligned} \left.\frac{d^2V(\phi_c)}{d\phi_c^2}\right|_{\phi_c=0} &= m^2 \\ \left.\frac{d^4V(\phi_c)}{d\phi_c^4}\right|_{\phi_c=0} &= \lambda \end{aligned} \tag{6.78}$$

then, the effective potential, up to one loop, can be rewritten (with an unimportant constant) to be

$$V(\phi_c) = V^0(\phi_c) + \frac{\hbar}{64\pi^2}\,(m_{eff}^4 \ln\frac{m_{eff}^2}{m^2} - \frac{3}{2}(m_{eff}^2-\frac{2}{3}m^2)^2) \tag{6.79}$$

It is worth noting here that we are considering a massive scalar theory and, consequently, the renormalization conditions in eq. (6.78) can be used without running into infrared divergences. In a massless theory,

on the other hand, one cannot use these conditions. In such a case, one defines the renormalized parameters at an arbitrary renormalization scale μ and the constants appearing in the expression for the effective potential would be different. Physical observables, however, can be shown to be independent of such an arbitrary energy scale.

We note here, for use in the next section, that the one loop effective potential in eq. (6.76) can also be written alternately by integrating out the fourth component of the Euclidean momentum, k_4, as (we would add an unimportant constant factor)

$$\begin{aligned} V^1(\phi_c) &= \frac{\hbar}{2}\int\frac{d^3k}{(2\pi)^3}\int\frac{dk_4}{2\pi}\,\ln(k_4^2+\omega^2) \\ &= \frac{\hbar}{2}\int\frac{d^3k}{(2\pi)^3}\int\frac{dk_4}{2\pi}\int_0^1 d\alpha\,\frac{\omega^2}{k_4^2+\alpha\omega^2} \end{aligned} \tag{6.80}$$

where we have defined

$$\omega^2 = \vec{k}^2 + m_{eff}^2 \tag{6.81}$$

The integrand in eq. (6.80) has poles at $k_4 = \pm i\sqrt{\alpha}\omega$. Consequently, if we interchange the order of the α and the k_4 integrations, we can evaluate it using the Residue theorem and it takes the form

$$V^1(\phi_c) = \frac{\hbar}{2}\int\frac{d^3k}{(2\pi)^3}\,\omega \tag{6.82}$$

Finally, we end this section by noting that we have calculated the one loop effective potential in a theory involving only a single scalar field. If the scalar field carries an internal index or if there are fermions and gauge fields present, the one loop contribution of these fields to the

scalar effective potential can also be calculated in a parallel manner and the general structure has the form

$$V^1 = \frac{\hbar}{64\pi^2} \mathrm{Tr}\ (\ M_S^4\ \ln M_S^2 - 4(M_F^\dagger M_F)^2\ \ln(M_F^\dagger M_F) + 3M_V^4\ \ln M_V^2\ ) \tag{6.83}$$

where we have defined the scalar, fermion and the gauge mass matrices from the classical action at zero momentum as (with ϕ_c constant and all other fields vanishing)

$$\begin{aligned} M_S^{2\,ij} &= -\left.\frac{\partial^2 S}{\partial\phi^i\partial\phi^j}\right|_{\phi=\phi_c} \\ M_F^{ab} &= \left.\frac{\partial^2 S}{\partial\bar{\psi}^a\partial\psi^b}\right|_{\phi=\phi_c} \\ \eta^{\mu\nu} M_V^{2\,\alpha\beta} &= \left.\frac{\partial^2 S}{\partial A_\mu^\alpha \partial A_\nu^\beta}\right|_{\phi=\phi_c} \end{aligned} \tag{6.84}$$

There are several things to note here. First, the fermion derivatives are defined to be left derivatives. Second, the fermion mass matrix will, in general, be complex depending on its coupling to the scalar fields and, consequently, the contribution of the fermions comes in the proper Hermitian combination (and with a negative sign because the integration over the fermion variables gives a determinant and not the inverse as is true for bosonic fields). Furthermore, in general, when we shift the scalar fields by ϕ_c, there would be mixing terms between the scalar and the gauge fields. However, such terms give vanishing contribution in the Landau gauge and the expression in eq. (6.84) corresponds to this choice of gauge. The effective potential, as we will discuss in more detail in the next chapter, is a gauge dependent quantity although the physical observables derived from it are gauge independent.

6.5 Symmetry Restoration

As we have noted in the beginning of this chapter, in physical systems such as a magnet where rotational symmetry is spontaneously broken, systems go through a phase transition when the temperature is raised and above a critical temperature the symmetry is restored. It is meaningful to ask whether a similar thing happens in the case of a quantum field theory with a spontaneously broken symmetry. Let us, for simplicity, consider the theory of eq. (6.1) or (6.3) with a spontaneous breakdown of a global symmetry. We have seen from eqs. (6.12)-(6.14) that a nontrivial vacuum exists leading to a spontaneous breaking of the symmetry only because the mass term has a negative sign. On the other hand, on physical grounds, we expect that if a particle moves in a hot medium, it must pick up a temperature dependent effective mass. In fact, we have already seen (see eq. (1.49)) that at one loop, the mass term in a quantum field theory becomes temperature dependent and that the temperature dependent corrections come with a positive sign. So, intuitively, it is clear that above a certain temperature, this temperature dependent correction will take over the tree level negative mass squared term and symmetry will be restored. Let us now study this phenomenon a bit more carefully. For simplicity and ease of comparison with the results of chapter 1, we will now set $\hbar = 1$. Second, since the effective potential, by definition, is a static quantity, we will use the imaginary time formalism for its evaluation although any of the real time formalisms can also be used with equal ease.

Let us consider the Lagrangian of eq. (6.3). We can, of course, evaluate the one loop effective potential in full generality follow-

ing the discussions of the previous section. However, let us simplify our calculation with the following observation. First, if a nontrivial vacuum does exist, then the solution will have the form of the second relation in eq. (6.14) and we can always choose $\chi = 0$. Therefore, it is meaningful and calculationally simple, for the purpose of evaluating the effective potential, to shift the fields as (σ_c = constant)

$$\sigma \to \sigma + \sigma_c \quad \chi \to \chi \tag{6.85}$$

With this shifting, the quadratic Lagrangian density in the quantum fields becomes (see eq. (6.3))

$$\mathcal{L}_{Qua} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{M_1^2}{2}\sigma^2 + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{M_2^2}{2}\chi^2 \tag{6.86}$$

where we have defined

$$\begin{aligned} M_1^2 &= (-m^2 + \frac{3\lambda}{2}\sigma_c^2) \\ M_2^2 &= (-m^2 + \frac{\lambda}{2}\sigma_c^2) \end{aligned} \tag{6.87}$$

It is now clear from the discussion of the last section (see eq. (6.75)) that the one loop correction is given by (remember $\hbar = 1$)

$$V^1(\sigma_c) = -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\left[\ln(-k^2 + M_1^2) + \ln(-k^2 + M_2^2)\right] \tag{6.88}$$

The only thing to remember in the case of the imaginary time formalism is that we have to rotate to Euclidean space with the energies taking discrete values (see eq. (1.29)) and the integration over the energy being replaced by a sum (see eq. (1.43)). Thus, the effective potential, in this

case, takes the form (with n taking both positive as well as negative integer values)

$$V^1(\sigma_c) = \frac{1}{2\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\,[\ln((\frac{2n\pi}{\beta})^2+\omega_1^2)+\ln((\frac{2n\pi}{\beta})^2+\omega_2^2)] \quad (6.89)$$

where we have defined

$$\omega_{1,2}^2 = \vec{k}^2 + M_{1,2}^2 \quad (6.90)$$

Let us, therefore, evaluate a generic expression of the form

$$\begin{aligned} I(M) &= \frac{1}{2\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\,\ln((\frac{2n\pi}{\beta})^2+\vec{k}^2+M^2) \\ &= \frac{1}{2\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\,\ln((\frac{2n\pi}{\beta})^2+\omega^2) \\ &= \int \frac{d^3k}{(2\pi)^3}\tilde{I}(\omega) \end{aligned} \quad (6.91)$$

with $\omega^2 = \vec{k}^2 + M^2$. Taking the derivative of $\tilde{I}(\omega)$ with respect to ω, we obtain

$$\frac{\partial\tilde{I}(\omega)}{\partial\omega} = \frac{1}{\beta}\sum_n \frac{\omega}{\omega^2+(\frac{2n\pi}{\beta})^2} \quad (6.92)$$

This sum can be easily evaluated using eq. (1.46) and gives

$$\begin{aligned} \frac{\partial\tilde{I}(\omega)}{\partial\omega} &= \left(\frac{1}{2}+\frac{1}{e^{\beta\omega}-1}\right) \\ \text{or,}\quad \tilde{I}(\omega) &= \frac{\omega}{2}+\frac{1}{\beta}\ln(1-e^{-\beta\omega}) \end{aligned} \quad (6.93)$$

Consequently, we can now write

$$I(M) = \int \frac{d^3k}{(2\pi)^3} \left(\frac{\omega}{2} + \frac{1}{\beta} \ln(1 - e^{-\beta\omega}) \right) \tag{6.94}$$

There are several things to note here. First, we see that for vanishing temperature (or $\beta \to \infty$), only the first term contributes and the expression reduces to eq. (6.82) (with $\hbar = 1$) as it should. The second term in the integrand, therefore, represents the true temperature corrections and we will denote this by $I^{(\beta)}$. Second, the temperature dependent part of the integral in eq. (6.94), $I^{(\beta)}$, cannot, in general, be evaluated in closed form, as is normally the case for most integrals in statistical mechanics. Even a high temperature expansion (β small) needs to be done carefully because some of the terms in the expansion become complex (since it corresponds to expanding in powers of M and M^2 can become negative when there is spontaneous symmetry breaking) signalling a breakdown of the approximation. The first few dominant terms can, however, be unambiguously calculated and is enough to give an approximate critical temperature for symmetry restoration and that is what we will calculate next. Let us note that

$$\begin{aligned} I^{(\beta)}(M)\Big|_{M^2=0} &= \frac{1}{2\pi^2\beta^4} \int_0^\infty dx\, x^2 \ln(1 - e^{-x}) = -\frac{\pi^2}{90\beta^4} \\ \frac{\partial I^{(\beta)}(M)}{\partial M^2}\Big|_{M^2=0} &= \frac{1}{4\pi^2\beta^2} \int_0^\infty dx\, \frac{x}{e^x - 1} = \frac{1}{24\beta^2} \end{aligned} \tag{6.95}$$

where both these integrals are standard integrals from statistical mechanics and we have used their values. This now shows that we can write

$$I^{(\beta)}(M) = -\frac{\pi^2}{90\beta^4} + \frac{M^2}{24\beta^2} + \cdots \tag{6.96}$$

Here "..." represent higher order terms that cannot be calculated reliably at this order of perturbation. We note here that the second term in eq. (6.96) has exactly the same structure as what we had calculated earlier in eq. (1.49).

Putting all of these into eq. (6.89), we see that the complete potential at one loop can be written as ($V^{(0)}$ represents the zero temperature potential including the one loop corrections.)

$$\begin{aligned} V(\sigma_c) &= V^{(0)}(\sigma_c) + V^{1\,(\beta)}(\sigma_c) \\ &= V^{(0)}(\sigma_c) - \frac{\pi^2}{45\beta^4} + \frac{M_1^2 + M_2^2}{24\beta^2} + \cdots \end{aligned} \tag{6.97}$$

As we have seen earlier (see eq. (6.78)), the second derivative of the zero temperature effective potential, $V^{(0)}$, would give the renormalized mass of the theory. Using this we obtain the temperature dependent effective mass of the theory to be

$$\begin{aligned} m^2(\beta) &= \left.\frac{d^2V^{(0)}}{d\sigma_c d\sigma_c}\right|_{\sigma_c=0} + \left.\frac{d^2V^{1\,(\beta)}}{d\sigma_c d\sigma_c}\right|_{\sigma_c=0} \\ &= -m^2 + \frac{\lambda}{6\beta^2} \end{aligned} \tag{6.98}$$

As we have discussed earlier, the nontrivial vacuum and the symmetry breaking disappear when the mass is nonnegative. This, therefore, defines a critical temperature for such a phase transition to be

$$\beta_c^2 = \frac{\lambda}{6m^2} \tag{6.99}$$

For temperatures larger than this critical temperature, the mass would be positive and there will be no symmetry breaking. The spontaneously

broken global symmetry of zero temperature is, therefore, restored above this critical temperature. Exactly similar analysis goes through for the case of a spontaneously broken local symmetry although the calculations are a bit more involved. The qualitative conclusions, however, remain the same.

6.6 Dynamical Symmetry Breaking

So far, we have discussed the case of symmetry breaking where a fundamental scalar field picks up a nonzero vacuum expectation value. Conventionally, such a symmetry breaking is known as spontaneous breaking of symmetry. There are situations, however, where a composite operator picks up a nonzero vacuum expectation value, in stead of a fundamental scalar, leading to the breaking of a symmetry. In such a case, one says that the symmetry is dynamically broken. The most popular of the composite operators whose vacuum expectation value one considers as breaking a symmetry is that of a pair of fermions. The classic examples are the pairing of fermions in superconductivity or the fermion condensation in the case of dynamical chiral symmetry breaking and so on. Dynamical symmetry breaking has a characteristic difference from spontaneous symmetry breaking in the following sense. In most cases of dynamical symmetry breaking, the nonzero value of the order parameter (vacuum expectation value of the composite operator) is associated with an anomaly of the theory such as the chiral anomaly. Anomalies, on the other hand, arise because of our inability to regularize the ultraviolet divergences of a quantum field theory maintaining certain symmetries. They really represent the ul-

traviolet behavior of a theory. However, as we have argued in chapter 2, the finite temperature effects do not change the ultraviolet behavior of a theory. So, intuitively, we expect temperature effects not to change the anomaly in a theory. Consequently, we would expect the dynamical breaking of a symmetry, associated with an anomaly,to be uninfluenced by temperature effects. In particular, unlike the case of the spontaneous breakdown of a symmetry, we do not expect a dynamically broken symmetry to be restored at high temperatures. (It is worth noting here that there are some models of dynamical symmetry breaking such as the Nambu-Jona-Lasinio model and QCD where the dynamical symmetry breaking is not associated with an anomaly. The behavior of these theories, as far as symmetry restoration is concerned, is more like the theories with spontaneous symmetry breaking and our discussion of this section does not apply to such theories.) In this section, we will show that this is indeed true with the simple example of the Schwinger model in $1+1$ dimensions.

The Schwinger model describes massless QED in $1+1$ dimensions and the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} - ieA_{\mu})\psi \qquad (6.100)$$

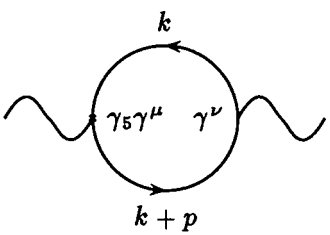
This is a gauge invariant Lagrangian density with a massless fermion and, in $1+1$ dimensions, the electromagnetic coupling has the dimensions of a mass. In addition to the local gauge invariance, the theory is also invariant under a global chiral transformation since the fermions are massless. However, as is well known, the chiral symmetry is anomalous in the quantum theory which leads the photon to become massive in the full theory. In this section, we will show that the chiral anomaly

is independent of temperature and, consequently, the dynamical mass generation is also temperature independent.

Let us note that the anomaly equation in this theory can be written as

$$\partial_\mu \langle T(j_5^\mu(x) j^\nu(0)) \rangle = G^\nu(x) \tag{6.101}$$

where the explicit form of G^ν is unimportant for our discussions, except for the fact that it is related to the chiral anomaly of the theory. It is clear that the question of temperature dependence of the anomaly is best studied within the framework of the real time formalism where propagators naturally separate into a temperature independent part and a temperature dependent part (see eqs. (2.36) and (2.45)). Diagrammatically, the anomaly term arises from a self-energy graph at one loop with a $\gamma_5\gamma^\mu$ vertex.



Thus, we see that at finite temperature, the anomaly of eq. (6.101) would have four contributions, namely,

$$I_0 = -e^2 \int \frac{d^2k}{(2\pi)^2} \, \mathrm{Tr}\, \gamma_5 \not{p} \not{k} \gamma^\nu (\not{k} + \not{p}) \frac{1}{k^2 (k+p)^2}$$

$$I_1 = -2\pi e^2 \int \frac{d^2k}{(2\pi)^2} \, \mathrm{Tr}\, \gamma_5 \not{p} \not{k} \gamma^\nu (\not{k} + \not{p}) \frac{1}{(k+p)^2} \frac{1}{e^{\beta|k^0|} + 1} \delta(k^2)$$

$$I_2 = -2\pi e^2 \int \frac{d^2k}{(2\pi)^2} \text{Tr}\, \gamma_5 \not{p} \not{k} \gamma^\nu (\not{k} + \not{p}) \frac{1}{k^2} \frac{1}{e^{\beta|k^0+p^0|}+1} \delta((k+p)^2)$$

$$I_3 = -(2\pi)^2 e^2 \int \frac{d^2k}{(2\pi)^2} \text{Tr}\, \gamma_5 \not{p} \not{k} \gamma^\nu (\not{k} + \not{p})$$

$$\times \frac{1}{e^{|k^0+p^0|}+1} \frac{1}{e^{\beta|k^0|}+1} \delta(k^2)\delta((k+p)^2) \qquad (6.102)$$

It is easy to show that I_1 and I_2 give the same contribution. This follows from the gamma matrix trace identity. First, we note that if we let $k \leftrightarrow -(k+p)$ in I_2 (these are finite integrals and hence there is no problem with the change of variables), it coincides with I_1 except for the trace part. However,

$$\begin{aligned} &\text{Tr}\, \gamma_5 \not{p} (\not{k} + \not{p}) \gamma^\nu \not{k} \\ &\quad = \text{Tr}\, \gamma_5 \not{p} \left[\not{k} \gamma^\nu (\not{k} + \not{p}) - \not{k} \gamma^\nu \not{p} + \not{p} \gamma^\nu \not{k} \right] \\ &\quad = \text{Tr}\, [\gamma_5 \not{p} \not{k} \gamma^\nu (\not{k} + \not{p}) + 2p^2 k^\nu \gamma_5] \\ &\quad = \text{Tr}\, \gamma_5 \not{p} \not{k} \gamma^\nu (\not{k} + \not{p}) \end{aligned} \qquad (6.103)$$

which is identical to the trace in I_1. The temperature dependent contributions come from I_1, I_2 and I_3. Let us note that if we use the two dimensional identity

$$\gamma_5 \gamma^\nu = \epsilon^{\nu\lambda} \gamma_\lambda$$

we can write

$$
\begin{aligned}
& I_1 + I_2 \\
& = -4\pi e^2 \epsilon^{\nu\lambda} \int \frac{d^2k}{(2\pi)^2} \, \mathrm{Tr}\, (\not{p}\not{k}\gamma_\lambda(\not{k}+\not{p})) \frac{1}{(k+p)^2} \frac{1}{e^{\beta|k^0|}+1} \delta(k^2) \\
& = -8\pi e^2 \epsilon^{\nu\lambda} \int \frac{d^2k}{(2\pi)^2} \, (k_\lambda (k+p)^2) \frac{1}{(k+p)^2} \frac{1}{e^{\beta|k^0|}+1} \delta(k^2) \\
& = -8\pi e^2 \epsilon^{\nu\lambda} \int \frac{d^2k}{(2\pi)^2} \, k_\lambda \, \frac{1}{e^{\beta|k^0|}+1} \delta(k^2) = 0
\end{aligned}
\tag{6.104}
$$

Here we have used the fact that k^2 vanishes because of the delta function in the intermediate steps and in the last step we have used the fact that the integral vanishes because the integrand is odd. It now follows trivially from the above calculation that (because the integrand in I_3 involves an additional $\delta((k+p)^2)$)

$$I_3 = 0 \tag{6.105}$$

This shows that the temperature dependent terms in the anomaly identically vanish and what is left is I_0 which is the zero temperature result. Consequently, we have shown that the anomaly is independent of temperature and so is the dynamical mass generation in this model.

6.7 References

Coleman, S and **E. Weinberg**, Phys. Rev. **D7** (1973) 1883.

Das, A, "Field Theory: A Path Integral Approach", World Scientific (1993).

Das, A and **A. Karev**, Phys. Rev. **D36** (1987) 623.

Dolan, L and **R. Jackiw**, Phys. Rev. **D9** (1974) 3320.

Goldstone, J, Nuovo Cimento, **19** (1961) 154.

Guralnik, G et al, in "Advances in Particle Physics", Ed. R. Cool and R. Marshak.

Jackiw, R, Phys. Rev. **D9** (1974) 1686.

Jona-Lasinio, G, Nuovo Cimento **34** (1964) 1790.

Nambu, Y, Phys. Lett. **26B** (1966) 626.

Nambu, Y and **G. Jona-Lasinio**, Phys. Rev. **122** (1961) 345.

Schwinger, J, Phys. Rev. **128** (1962) 2425.

Weinberg, S, Phys. Rev. **D9** (1974) 3357.

Chapter 7

Nielsen Identities

As we have seen in the last chapter, the effective potential plays a crucial role in studying symmetry breaking and restoration in a quantum field theory. A theory may already display spontaneous breakdown of symmetry at the tree level and yet it is useful to examine the behavior of the effective potential at higher loops for various reasons. First, of course, we would like to know whether the tree level symmetry breaking is stable under radiative corrections. Second, sometimes in a gauge theory, the Higgs potential may have a larger symmetry than is present in the full theory. In such a case, the tree level breaking of this larger symmetry of the Higgs potential may lead to the appearance of massless "pseudo-Goldstone" bosons in the theory which may become massive due to radiative corrections. More importantly, sometimes a symmetry may not be broken at the tree level and yet a breaking may

be induced when radiative corrections are included. For all these reasons, as well as to determine if there is a restoration of a symmetry, it is useful to study the behavior of the effective potential in a quantum field theory including radiative corrections. However, a serious conceptual problem arises in this endeavor in that the effective potential, in a gauge theory, is not a gauge independent quantity. And, therefore, it is not clear *a priori* how much of the analysis resulting from a study of the effective potential is physically relevant or can be taken seriously. In this chapter, we will discuss this issue in some detail.

7.1 Gauge Problem

To begin with, let us recall that the effective potential is obtained from the effective action of the theory (see eqs.(6.55) and (6.56)) when the scalar fields take a constant value (with all other fields vanishing). Therefore, it is a generator of Greens functions with zero external momentum which are by definition off-shell. Off-shell Greens functions, on the other hand, need not be gauge independent and that shows that, in general, the effective potential, in a gauge theory, is not gauge independent. However, we also recognize (from the argument following eq. (6.11)) that the value of the effective potential at its minimum represents the ground state energy which is a physical quantity and we expect this to be gauge independent. Thus, we already see from this that the gauge independence of the effective potential needs to be studied more carefully.

To see that the effective potential can, in fact, become gauge dependent, even at the tree level, it is best to look at the Abelian Higgs model in the R_ξ gauge (see eq. (6.33)). The Lagrangian density, before shifting the fields, becomes in this gauge (it is the same as the sum of eq. (6.23) and eq. (6.33) only if we remember that the fields are not yet shifted),

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{m^2}{2}(\sigma^2+\chi^2) \\ &\quad + e(\chi\partial^\mu\sigma - \sigma\partial^\mu\chi)A_\mu + \frac{e^2}{2}(\sigma^2+\chi^2)A_\mu A^\mu - \frac{\lambda}{8}(\sigma^2+\chi^2)^2 \\ &\quad - \frac{1}{2\xi}(\partial_\mu A^\mu + \xi e v\chi)^2 + \partial_\mu\bar{c}\partial^\mu c - \xi e^2 v\sigma\bar{c}c \qquad (7.1)\end{aligned}$$

Here, we have used $v = \langle\sigma\rangle$. It is clear now that for constant values of the σ and the χ fields, with all other fields set equal to zero, the tree level potential becomes

$$V^0(\sigma,\chi) = -\frac{m^2}{2}(\sigma^2+\chi^2) + \frac{\lambda}{8}(\sigma^2+\chi^2)^2 - \frac{\xi(ev)^2}{2}\chi^2 \qquad (7.2)$$

This shows explicitly that even the tree level potential is gauge dependent in this simple model. (We have, of course, also noted earlier, in eq. (6.34), that the tree level mass for the χ field in the R_ξ gauge is gauge dependent.)

The argument above holds for any generalized R_ξ gauge as well and has led to a lot of confusion in the literature. Therefore, let us understand the structure of the tree level potential first. There are several objections to the structure of the theory in eqs. (7.1) and the potential in eq. (7.2). First, there is a conceptual question. We have

included in the gauge fixing term (and, therefore, in the tree level potential) the vacuum expectation value of the scalar field even before minimizing the potential and so, naturally, the question arises as to whether the whole procedure is consistent. In fact, if one tries to minimize the potential in the R_ξ gauge (generalized or not), one finds spurious gauge dependent solutions. Second, for various reasons, it has been argued in the literature that the R_ξ gauge may not be a good gauge for the study of effective potential. The answer to both these questions, as we will see below, is that the choice of the gauge fixing term does not influence the minimization of the potential and that the R_ξ gauge is perfectly fine as far as the study of the effective potential is concerned. This is best seen if we use an auxiliary field to write the gauge fixing Lagrangian density (see eqs. (4.19) and (4.20)). Thus, to make the discussion clear, we will deviate from the standard description of the subject in the literature and use the auxiliary field formulation. We note that with the auxiliary field, $\mathcal{L}_{GF}$ takes the form (see eq. (4.20))

$$\mathcal{L}_{GF} = \frac{\xi}{2}F^2 + \partial_\mu F A^\mu - \xi e v F \chi \tag{7.3}$$

which upon using the equation for the auxiliary field

$$F = \frac{1}{\xi}(\partial_\mu A^\mu + \xi e v \chi) \tag{7.4}$$

would lead to the gauge fixing Lagrangian density in the R_ξ gauge of eq. (7.1) if we ignore a total divergence term.

The important point to recall here is the discussion of chapter 4, namely, eq. (4.28), which says that the BRST variation of the anti-ghost field is the auxiliary field and since the BRST transforma-

tions are nilpotent, the auxiliary field remains invariant under a BRST transformation. This implies that

$$\begin{aligned} [Q_{BRST}, \bar{c}]_+ &= -F \\ [Q_{BRST}, F] &= 0 \end{aligned} \tag{7.5}$$

Combining this with the physical state condition of eq. (4.34), we obtain now (from the first relation in eq. (7.5)) that in the true vacuum of the theory

$$\langle 0|F|0\rangle = 0 \tag{7.6}$$

This shows that the gauge fixing Lagrangian gives a vanishing contribution to the energy of the true vacuum and hence cannot influence the minimization of the potential. (See eqs. (7.4) and (7.1). Incidentally, we note here that a bad choice of the gauge fixing Lagrangian may force that at the minimum, the gauge field takes a nonlocal, space-time dependent value for eq. (7.6) to be true, but the gauge fixing Lagrangian would still contribute trivially to the minimum of the potential. What we have in mind here is as follows. Suppose, we want to choose as a minimum $\langle\sigma\rangle \neq 0$ and $\langle\chi\rangle = 0$. In such a case, if we had chosen a gauge fixing Lagrangian of the form

$$\mathcal{L}_{GF} = \frac{\xi}{2}F^2 + \partial_\mu F A^\mu + \beta F\sigma$$

with β an arbitrary constant parameter, then the equation for the auxiliary field would yield

$$\xi F = \partial_\mu A^\mu - \beta\sigma$$

It is clear now that, at the minimum, for $\langle F\rangle$ to vanish, we must have

$$\langle \partial_\mu A^\mu \rangle = \beta \langle \sigma \rangle \neq 0$$

And, that would, for consistency require that the vacuum expectation value of the gauge field cannot be set equal to zero.)

In fact, from eq. (7.5) and the physical state condition (4.34), it is easy to see that

$$\langle 0|F(x_1)F(x_2)\cdots F(x_n)|0\rangle = 0 \tag{7.7}$$

This, in particular, implies that the two point correlation involving the auxiliary fields and, therefore, the propagator for the F field vanishes

$$\langle 0|F(x_1)F(x_2)|0\rangle = 0 \tag{7.8}$$

This can, in fact, be explicitly checked. We see from eqs. (7.1) and (7.3) that there is mixing between the A_μ and F fields. The two point function for the

$$\begin{pmatrix} F \\ A_\mu \end{pmatrix}$$

system has the form

$$\begin{pmatrix} \xi & ik^\mu \\ -ik^\nu & -(\eta^{\mu\nu}k^2 - k^\mu k^\nu) \end{pmatrix} \tag{7.9}$$

The inverse of this matrix can be worked out in a straightforward manner and has the form

$$\begin{pmatrix} 0 & \frac{ik_\nu}{k^2} \\ -\frac{ik_\mu}{k^2} & -\frac{1}{k^2}(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}) \end{pmatrix} \tag{7.10}$$

This shows explicitly that the F propagator vanishes. This is consistent with eq. (7.7) or (7.8).

We note from eq. (7.2) that while the potential is explicitly gauge dependent, its value at the minimum (see eq. (6.15)), namely,

$$\langle\sigma\rangle = v \qquad \langle\chi\rangle = 0$$

is independent of the gauge fixing parameter. It can be seen more generally as follows. From eqs. (7.1) and (7.3), we see that

$$\begin{aligned}
\langle 0|\frac{\partial S}{\partial \xi}|0\rangle &= \int d^4x\, \langle 0|(\frac{1}{2}F^2 - evF\chi - e^2v\sigma\bar{c}c)|0\rangle \\
&= \int d^4x\, \langle 0| - \delta_{BRST}[\bar{c}(\frac{1}{2}F - ev\chi)]|0\rangle \\
&= -\int d^4x\, \langle 0| \left[Q_{BRST}, \bar{c}(\frac{1}{2}F - ev\chi)\right]_+ |0\rangle \\
&= 0 \qquad (7.11)
\end{aligned}$$

where S represents the complete action of our theory and where we have used the physical state condition of eq. (4.34) (The BRST variation is without the parameter of transformation and in taking out the parameter of transformation, we have to remember that it anti-commutes with the anti-ghost field, $\bar{c}$. The explicit BRST transformations are given later in eq. (7.13).). This shows that the value of the potential at its minimum is gauge independent as it should be since it represents the ground state energy which is an observable quantity.

The discussion so far makes it very clear that since the effective potential is a gauge dependent quantity, we should determine systematically which of the quantities derived from it will be gauge independent. It is worth noting here that with the use of the auxiliary fields, one can carry out the discussion of Abelian as well as non-Abelian gauge theories with equal ease. For simplicity, however, we will restrict ourselves to the example of the Abelian Higgs model for the rest of this chapter.

7.2 Identities at $T = 0$

Nielsen identities are identities describing the gauge dependence of the effective potential. As we have seen in chapter 4, quantities which are not gauge invariant satisfy Ward Identities following from the BRST invariance of the theory. Ward Identities, basically, express how non-gauge invariant quantities would change under a gauge transformation or a change of gauge. Therefore, we see that since the effective potential is not gauge independent, the relations (identities) that it must satisfy are best described within the framework of BRST invariance. And in trying to derive these, we will not follow the conventional description, rather, we will use the auxiliary field formalism which is much simpler.

Let us, for simplicity, consider the Abelian Higgs model in the R_ξ gauge (before shifting) although, as we have tried to emphasize earlier, with the auxiliary fields, non-Abelian theories are described with equal ease as well. The Lagrangian density is (see eq. (7.1))

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{m^2}{2}(\sigma^2+\chi^2) \\ &+e(\chi\partial^\mu\sigma - \sigma\partial^\mu\chi)A_\mu + \frac{e^2}{2}(\sigma^2+\chi^2)A_\mu A^\mu - \frac{\lambda}{8}(\sigma^2+\chi^2)^2 \\ &+\frac{\xi}{2}F^2 + \partial_\mu F A^\mu - \xi e v F\chi + \partial_\mu\bar{c}\partial^\mu c - \xi e^2 v\sigma\bar{c}c \qquad (7.12)\end{aligned}$$

Following our discussion of the derivation of Ward Identities in chapter 4, we now introduce sources for the various fields of our theory as well as their BRST variations. However, since this is a simple Abelian theory, the BRST variations are much simpler (see eqs. (4.28) and (6.25))

$$\begin{aligned}\delta_{BRST}A_\mu &= \omega\partial_\mu c \\ \delta_{BRST}\sigma &= -\omega e\chi c \\ \delta_{BRST}\chi &= \omega e\sigma c \\ \delta_{BRST}c &= 0 \\ \delta_{BRST}\bar{c} &= -\omega F \\ \delta_{BRST}F &= 0 \qquad (7.13)\end{aligned}$$

Consequently, the only two composite variations for which we have to add any sources are $\delta_{BRST}\sigma$ and $\delta_{BRST}\chi$.

Thus, the source Lagrangian density takes the form

$$\begin{aligned}\mathcal{L}_{Source} &= J_\mu A^\mu + JF + K\sigma + L\chi + i(\bar{\eta}c - \bar{c}\eta) \\ &+M(-e\chi c) + N(e\sigma c) + H[\bar{c}(\frac{1}{2}F - ev\chi)] \qquad (7.14)\end{aligned}$$

where, in addition to the usual sources used in the derivation of the Ward Identities (see eq. (4.37)), we have added an extra source, H, in the last term. Comparing this last term with eq. (7.11), it is clear that the variation of this last term under a BRST transformation would yield a term which would give the gauge dependence of the effective potential. (We note here, parenthetically, that, with the auxiliary field, the gauge dependence is obtained exactly from the BRST variation of this term and we do not have to worry about a divergence or the ghost equations of motion etc., as is normally the case in the literature.) Thus, with the two Lagrangian densities in eqs. (7.12) and (7.14) together, we can write the generating functional for the system as

$$Z[J] = e^{iW[J]} = \int \mathcal{D}\phi \, e^{i\int d^4x \, (\mathcal{L}+\mathcal{L}_{Source})} \tag{7.15}$$

where we have denoted all the sources and fields generically by J and ϕ respectively. Let us next shift all the fields under the integral by their BRST variations holding the sources fixed. In such a case, as we have noted earlier in eqs. (4.40)-(4.41),

$$\int \mathcal{D}\phi \int d^4x \, [J_\mu \delta_{BRST} A^\mu + K\delta_{BRST}\sigma + L\delta_{BRST}\chi - i(\delta_{BRST}\bar{c})\eta$$
$$+ H\delta_{BRST}(\bar{c}(\frac{1}{2}F - ev\chi))] \times e^{i\int d^4x \, (\mathcal{L}+\mathcal{L}_{Source})} = 0 \tag{7.16}$$

We can rearrange the terms and note from the BRST variations in eq. (7.13) that we can write this also as

$$\int \mathcal{D}\phi \int d^4x \, H(x)\frac{\partial \mathcal{L}}{\partial \xi} \, e^{i\int d^4x \, (\mathcal{L}+\mathcal{L}_{Source})}$$
$$= -\int d^4x \, (iJ_\mu \partial^\mu \frac{\delta W}{\delta \bar{\eta}} + K\frac{\delta W}{\delta M} + L\frac{\delta W}{\delta N} + i\frac{\delta W}{\delta J}\eta) \tag{7.17}$$

where we have also used our earlier observation in eq. (7.11).

We see from eq. (7.17) that if we take the functional derivative of the left hand side with respect to $H(x)$ and integrate over x, we will obtain the gauge dependence of the generating functional, namely, eq. (7.17), in such a case, would give

$$\frac{\partial W}{\partial \xi} = -\int d^4x\, d^4y \left(iJ_\mu(y)\partial_y^\mu \frac{\delta^2 W}{\delta H(x)\delta\bar{\eta}(y)} + K(y)\frac{\delta^2 W}{\delta H(x)\delta M(y)} + L(y)\frac{\delta^2 W}{\delta H(x)\delta N(y)} + i\frac{\delta^2 W}{\delta H(x)\delta J(y)}\eta(y)\right) \tag{7.18}$$

Here we have assumed that all the sources are linearly independent. Furthermore, we note that if we now set all the sources equal to zero, we will obtain a relation describing the gauge dependence of the effective potential. (Setting all the sources equal to zero is equivalent to putting all the classical fields except the scalar fields to zero.)

A more direct way to obtain a relation for the effective potential is to go to the effective action. Thus, as in chapter 4, we define a Legendre transformation with respect to the sources J_μ, J, K, L, η and $\bar{\eta}$ as

$$\Gamma = W - \int d^4x\, (J_\mu A^\mu + JF - K\sigma + L\chi + i(\bar{\eta}c - \bar{c}\eta)) \tag{7.19}$$

where, as before, the fields appearing on the right hand side are the classical fields although we do not explicitly write it. Since the Legendre transformation does not involve the sources M, N and H, it follows now that

$$\frac{\delta\Gamma}{\delta M} = \frac{\delta W}{\delta M} \quad , \quad \frac{\delta\Gamma}{\delta N} = \frac{\delta W}{\delta N}$$
$$\frac{\delta\Gamma}{\delta H} = \frac{\delta W}{\delta H} \quad , \quad \frac{\partial\Gamma}{\partial\xi} = \frac{\partial W}{\partial\xi} \tag{7.20}$$

while for the derivatives with respect to the classical fields, we have

$$\frac{\delta\Gamma}{\delta A^\mu} = -J_\mu \quad , \quad \frac{\delta\Gamma}{\delta F} = -J$$
$$\frac{\delta\Gamma}{\delta\sigma} = -K \quad , \quad \frac{\delta\Gamma}{\delta\chi} = -L$$
$$\frac{\delta\Gamma}{\delta c} = i\bar{\eta} \quad , \quad \frac{\delta\Gamma}{\delta\bar{c}} = i\eta \tag{7.21}$$

With these relations, eq. (7.18) can now be written as

$$\frac{\partial\Gamma}{\partial\xi} = \int d^4x\, d^4y \left(\frac{\delta\Gamma}{\delta A^\mu(y)} \partial_y^\mu \frac{\delta c(y)}{\delta H(x)} + \frac{\delta\Gamma}{\delta\sigma(y)} \frac{\delta^2\Gamma}{\delta H(x)\delta M(y)} \right.$$
$$\left. + \frac{\delta\Gamma}{\delta\chi(y)} \frac{\delta^2\Gamma}{\delta H(x)\delta N(y)} + \frac{\delta F(y)}{\delta H(x)} \frac{\delta\Gamma}{\delta\bar{c}(y)} \right) \tag{7.22}$$

If we now set all the classical fields to zero except σ (*i.e.*, we are assuming a particular form of the minimum consistent with our choice of the gauge fixing Lagrangian, but the arguments do not depend on the choice of the minimum at all as we have emphasized before), then, the left hand side of eq. (7.22) represents the derivative of the effective potential with respect to the gauge fixing parameter up to a proportionality factor. On the right hand side, the first term vanishes because of Lorentz invariance (namely, there is no one point function for the gauge field) and the last term vanishes because of ghost number conservation (there is no one point function for the ghost field). The

other two terms, in principle, survive. But, if we look at the structure of $\mathcal{L}_{Source}$ in eq. (7.14), it is clear that the radiative corrections can lead to a two point function involving H and M, but not for H and N. Therefore, in this limit, eq. (7.22) becomes

$$\frac{\partial V_{eff}}{\partial \xi} = -C(\sigma, \xi)\frac{\partial V_{eff}}{\partial \sigma} \tag{7.23}$$

where we have used the translational invariance of the two point function to define

$$C(\sigma, \xi) = -\int d^4x \, \frac{\delta^2 \Gamma}{\delta H(x) \delta M(0)} \tag{7.24}$$

with all fields set to zero except σ. This is the essence of the Master identity due to Nielsen which can be thought of as a Ward Identity for the effective potential.

To understand the meaning of eq. (7.23), we note that even though the effective potential is gauge dependent, at the minimum where

$$\frac{\partial V_{eff}}{\partial \sigma} = 0$$

we have from eq. (7.23)

$$\frac{\partial V_{eff}}{\partial \xi}\Big|_{\min} = 0 \tag{7.25}$$

This shows that the value of the effective potential at its minimum is gauge independent as it should be considering that it represents the ground state energy. (Incidentally, we note here that not only is the value of the potential gauge independent at its minimum, it is also gauge independent at all its extrema. This, then, would imply that

the difference in the values of the potential at the false vacuum and the true one, say for example, in the case of the double well potential would also be gauge independent. This height is quite relevant in the study of various physical phenomena such as tunnelling, rate of decay of the false vacuum or some questions related to phase transitions and it is reassuring to know that this does not depend on the gauge fixing parameter.) Furthermore, as a partial differential equation, eq. (7.23) expresses the fact that the effective potential V_{eff} is stationary (with respect to variations in ξ) along the characteristics given by

$$\frac{\partial \sigma}{\partial \xi} = C(\sigma, \xi) \tag{7.26}$$

and that eq. (7.23) can be expressed, in such a case, as

$$\frac{dV_{eff}}{d\xi} = 0 \tag{7.27}$$

Namely, we see from eq. (7.26) that the expectation value of the field is gauge dependent leading to both an implicit and an explicit dependence of the effective potential on the gauge fixing parameter. However, the two contributions cancel each other along the characteristics leading to the fact that the value of the effective potential is truly independent of the gauge fixing parameter along the characteristics (*i.e.*, when we consider the total gauge dependence both explicit and implicit). This is true even if we are not at a minimum of the effective potential.

It is interesting to note that the expectation value of the scalar field, although gauge independent at the tree level, can become gauge dependent at higher orders. We could have derived, directly, the

gauge dependence of this from eq. (7.18). In fact, noting that (see, for example, eq. (4.39) or (6.39))

$$\frac{\delta W}{\delta K(x)} = \sigma(x)$$

we see that if we take the functional derivative of eq. (7.18) with respect to $K(x)$ and set all the sources equal to zero (which is also equivalent to setting all the fields to zero except $\sigma(x) = \sigma$, a constant), we obtain

$$\begin{aligned} \frac{\partial \sigma}{\partial \xi} &= -\int d^4x \, \frac{\delta^2 W}{\delta H(x) \delta M(0)} \\ &= -\int d^4x \, \frac{\delta^2 \Gamma}{\delta H(x) \delta M(0)} \\ &= C(\sigma, \xi) \end{aligned} \tag{7.28}$$

where we have used the identification in eq. (7.24) in the last step. This result is quite interesting in the following sense. We have argued before that we can think of the vacuum expectation value of the scalar field as an order parameter of symmetry breaking. However, we find that this is a gauge dependent quantity. It is not clear, therefore, whether we can use this quantity to study phenomena such as symmetry restoration and the critical temperature associated with the phase transition. Furthermore, it is not clear whether physical quantities, such as the mass, derived from the effective potential will be independent of the gauge fixing parameter (We expect that it should be). All such questions can, in fact, be studied starting from the Master equation in (7.23).

Let us look at the gauge independence of the Higgs mass (mass of the σ field) for simplicity. The discussion for other physical

quantities can be carried out along the same lines. First, we note that there are two kinds of masses that we can define in a field theory. First, there is the mass parameter obtained from the second derivative of the effective potential. We can think of this as the two point function at vanishing momentum. The physical mass, on the other hand, is obtained from the pole of the propagator which we can also think of as the value of p^2 (momentum squared) for which the two point function vanishes. The two masses agree only at the tree level. However, to obtain the physical mass, we should really look at the inverse propagator or the two point function. Keeping this in mind, we take the second functional derivative of eq. (7.22) with respect to $\sigma(x)$ and $\sigma(y)$ and then set all the classical fields to zero except σ which takes a constant value. In this case, we obtain, from eq. (7.12),

$$\begin{aligned}\frac{\partial}{\partial\xi}\frac{\delta^2\Gamma}{\delta\sigma(x)\delta\sigma(y)}\bigg|_{\rm min} &= \int d^4\omega\, d^4z \left[\frac{\delta^2\Gamma}{\delta\sigma(z)\delta\sigma(x)}\frac{\delta^3\Gamma}{\delta H(\omega)\delta M(z)\delta\sigma(y)}\right.\\ &\quad + \frac{\delta^2\Gamma}{\delta\sigma(z)\delta\sigma(y)}\frac{\delta^3\Gamma}{\delta H(\omega)\delta M(z)\delta\sigma(x)}\\ &\quad \left. + \frac{\delta^3\Gamma}{\delta\sigma(z)\delta\sigma(x)\delta\sigma(y)}\frac{\delta^2\Gamma}{\delta H(\omega)\delta M(z)}\right]_{\rm min} \end{aligned} \tag{7.29}$$

Here the "min" stands for setting all fields to zero except σ to a constant giving the minimum of the effective action.

We note here that, by definition,

$$\frac{\delta^2\Gamma}{\delta\sigma(x)\delta\sigma(y)}\bigg|_{\rm min} = G^{-1}(x-y;\sigma) \tag{7.30}$$

where G represents the renormalized propagator for the σ field. Furthermore, we are interested in the form of the equation (7.30) at points where the inverse propagators vanish. It is clear that, at such points, the first two terms on the right hand side of eq. (7.29) would vanish. Therefore, in the momentum space, eq. (7.29) would take the form (for vanishing two point function)

$$\frac{\partial G^{-1}(p^2;\sigma)}{\partial \xi} = -C(\sigma,\xi)\frac{\partial G^{-1}(p^2;\sigma)}{\partial \sigma} \tag{7.31}$$

where we have used the definition in eq. (7.24). It is clear now that if we represent

$$G^{-1}(p^2;\sigma) = (p^2 - m_P^2(\sigma,\xi)) \tag{7.32}$$

where m_P represents the physical mass corresponding to the pole of the propagator, eq. (7.31) would take the form

$$\begin{aligned} \frac{\partial m_P^2}{\partial \xi} &= -C(\sigma,\xi)\frac{\partial m_P^2}{\partial \sigma} \\ \text{or,}\quad \frac{dm_P^2}{d\xi} &= 0 \end{aligned} \tag{7.33}$$

along the characteristics (see eq. (7.26)). Once again, following our earlier discussion, this shows that the physical Higgs mass is independent of the gauge fixing parameter along any characteristics. The gauge independence of other physical quantities can also be derived following the same line of reasoning.

Next, we come to the question of symmetry restoration. We have seen, in chapter 6, that a spontaneously broken symmetry is restored when the scalar vacuum expectation value vanishes. We have also

seen that this happens when the mass parameter (the second derivative of the effective potential) vanishes. But what we have seen so far, from the discussions in this chapter, is that the vacuum expectation value of the scalar field (the "order" parameter) is gauge dependent. A natural question that arises, then, is whether such an analysis is meaningful at all. To examine this question, we go to the Master equation (7.23) and take the second derivative with respect to σ which gives, at the minimum of the potential,

$$\frac{\partial}{\partial \xi}\frac{\partial^2 V_{eff}}{\partial \sigma^2} + C(\sigma,\xi)\frac{\partial}{\partial \sigma}\frac{\partial^2 V_{eff}}{\partial \sigma^2} + 2\frac{\partial C}{\partial \sigma}\frac{\partial^2 V_{eff}}{\partial \sigma^2} = 0 \tag{7.34}$$

This, as we can clearly see, does not have a gauge independent structure because of the last term (see eq. (7.27) or (7.33)). But we note that if the second derivative of the effective potential vanishes (*i.e.* when the mass parameter in the theory vanishes) the equation takes a gauge independent form

$$\frac{\partial}{\partial \xi}\frac{\partial^2 V_{eff}}{\partial \sigma^2} + C(\sigma,\xi)\frac{\partial}{\partial \sigma}\frac{\partial^2 V_{eff}}{\partial \sigma^2} = 0 \tag{7.35}$$

This shows that even if the scalar vacuum expectation value is gauge dependent, we can still study symmetry restoration and calculate a critical temperature associated with such a phase transition in a gauge independent fashion by analyzing when the mass parameter of the theory vanishes.

Finally, we note here without going into details that the gauge fixing Lagrangian density, as we have seen, does not lead to any observable properties. From eq. (7.12), we see that the gauge fixing Lagrangian density depends on another parameter v and so we conclude

that the physical quantities cannot depend on this parameter either. In fact, following the line of argument that led to identities involving ξ dependence, we can also derive identities involving v dependence which would also show that physical observables will be independent of the choice of this parameter.

7.3 Verification of Identities

The interesting thing about the Nielsen identities (7.23) is that the function $C(\sigma, \xi)$ has a closed form expression given in eq. (7.24). Therefore, the identities can be verified order by order in perturbation theory. Thus, for example, we have already seen that at the tree level, there is no gauge dependence of the potential and to order one loop, we can write the relation in eq. (7.23) as

$$\frac{\partial V^1}{\partial \xi} + C^0 \frac{\partial V^1}{\partial \sigma} + C^1 \frac{\partial V^0}{\partial \sigma} = 0 \tag{7.36}$$

where the superscript denotes the order at which various quantities are evaluated. However, looking at the source Lagrangian in eq. (7.14), we note that at the tree level, there is no mixing between the sources H and M and consequently, from eq. (7.24) we conclude that

$$C^0(\sigma, \xi) = 0 \tag{7.37}$$

Thus, at one loop order, the Nielsen identities take the simple form

$$\frac{\partial V^1}{\partial \xi} + C^1 \frac{\partial V^0}{\partial \sigma} = 0 \tag{7.38}$$

To evaluate the effective potential at one loop, we follow the discussion of chapter 6. In this simple case, with our choice of symmetry breaking, we only need to shift

$$\sigma \longrightarrow \sigma + \sigma_c$$

For simplicity, we will also identify v in the Lagrangian density (7.12) with σ_c although, in general, one can leave this as an arbitrary parameter. Thus, we obtain from the Lagrangian density (7.12) (see also eq. (7.2))

$$V^0(\sigma_c) = -\frac{m^2}{2}\sigma_c^2 + \frac{\lambda}{8}\sigma_c^4$$

so that, we have

$$\frac{\partial V^0}{\partial \sigma_c} = -\sigma_c(m^2 - \frac{\lambda}{2}\sigma_c^2) \tag{7.39}$$

For the evaluation of the one loop effective potential, we need the part of the Lagrangian density in eq. (7.12) which is quadratic in the quantum fields after shifting and this has the form

$$\begin{aligned}
\mathcal{L}_Q &= \frac{1}{2}A_\mu \left(\eta^{\mu\nu}(\Box + e^2\sigma_c^2) - (1 - \frac{1}{\xi})\partial^\mu\partial^\nu\right) A_\nu \\
&+\frac{1}{2}\sigma(-\Box + m^2 - \frac{3\lambda}{2}\sigma_c^2)\sigma + \frac{1}{2}\chi(-\Box + m^2 - (\frac{\lambda}{2} + \xi e^2)\sigma_c^2)\chi \\
&+\frac{\xi}{2}F^2 - \bar{c}(\Box + \xi e^2\sigma_c^2)c
\end{aligned} \tag{7.40}$$

In writing the above Lagrangian, we have diagonalised terms involving mixing between the F, A_μ and the χ fields by redefining

$$F \longrightarrow F + \frac{1}{\xi}(\partial_\mu A^\mu + \xi e \sigma_c \chi) \tag{7.41}$$

which makes the calculation simpler. Denoting the two point functions for the gauge field, the scalar fields and the ghost fields by $G^{-1\,\mu\nu}$, G_σ^{-1}, G_χ^{-1}, G_F^{-1} and S^{-1} respectively, we note from our discussions in chapter 6 that the one loop effective potential can be written as ($\hbar = 1$)

$$\begin{aligned} V^1 &= -\frac{i}{2}\Omega^{-1}\mathrm{Tr}\left[\ln G^{-1\,\mu\nu} + \ln G_\sigma^{-1} + \ln G_\chi^{-1} + \ln G_F^{-1}\right. \\ &\qquad \left. -2\ln S^{-1}\right] \end{aligned} \tag{7.42}$$

Here the ghost contribution is different from the others because, as we have discussed earlier, the ghost fields are anti-commuting and are more like fermions leading to the negative sign and the factor of two arises because they are complex. We note here that G_σ^{-1} does not involve the gauge fixing parameter and, therefore, would not contribute to $\frac{\partial V^1}{\partial \xi}$. Consequently, we have

$$\begin{aligned} \frac{\partial V^1}{\partial \xi} &= -\frac{i}{2}\Omega^{-1}\mathrm{Tr}\left[G_{\mu\nu}\frac{\partial G^{-1\,\nu\mu}}{\partial \xi} + G_\chi \frac{\partial G_\chi^{-1}}{\partial \xi} + G_F \frac{\partial G_F^{-1}}{\partial \xi}\right. \\ &\qquad \left. -2S\frac{\partial S^{-1}}{\partial \xi}\right] \end{aligned} \tag{7.43}$$

The relevant functions can be read out from eq. (7.40) to have the form, in momentum space,

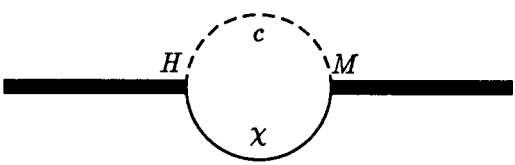
$$\begin{aligned} G^{-1\,\mu\nu} &= \left(\eta^{\mu\nu}(-p^2 + e^2\sigma_c^2) + (1 - \frac{1}{\xi})p^\mu p^\nu\right) \\ G_{\mu\nu} &= \frac{1}{-p^2 + e^2\sigma_c^2}\left(\eta_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2 - \xi e^2 \sigma_c^2}\right) \end{aligned}$$

$$\begin{aligned} G_\chi^{-1} &= (p^2 + m^2 - (\frac{\lambda}{2} + \xi e^2)\sigma_c^2) \\ G_F^{-1} &= \xi \\ S^{-1} &= (p^2 - \xi e^2 \sigma_c^2) \end{aligned} \tag{7.44}$$

Using these relations and following the discussion in chapter 6, we can now simplify eq. (7.43) to obtain

$$\frac{\partial V^1}{\partial \xi} = -\frac{i}{2}\int \frac{d^4p}{(2\pi)^4} \frac{e^2\sigma_c^2(m^2 - \frac{\lambda}{2}\sigma_c^2)}{(p^2 - \xi e^2\sigma_c^2)(p^2 + m^2 - (\frac{\lambda}{2} + \xi e^2)\sigma_c^2)} \tag{7.45}$$

The only thing that remains to be evaluated is C^1. We note that, at one loop, the Feynman diagram which leads to C^1 (*i.e.* the two point function corresponding to the sources H and M) has the following form



We already know the propagators for the χ field and the ghost field from eq. (7.44). We also see that under the shifting, eq. (7.41), the source Lagrangian density in eq. (7.14) takes the form

$$\begin{aligned} \mathcal{L}_{Source} \rightarrow\ & J_\mu A^\mu + JF + K\sigma + L\chi + i(\bar{\eta}c - \bar{c}\eta) + M(-e\chi c) \\ & + N(e\sigma c) + \frac{1}{2}H[\bar{c}(F + \frac{1}{\xi}\partial_\mu A^\mu - ev\chi)] \end{aligned} \tag{7.46}$$

Thus, the calculation of C^1 is now straightforward (taking into account the factors of i coming from the vertices, the propagators and the definition of the amplitude)

$$\begin{aligned} C^1 &= -\frac{i}{2}e^2\sigma_c \int \frac{d^4p}{(2\pi)^4}\, S(p)G_\chi(-p) \\ &= -\frac{i}{2}e^2\sigma_c \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - \xi e^2\sigma_c^2)(p^2 + m^2 - (\frac{\lambda}{2} + \xi e^2)\sigma_c^2)} \end{aligned} \tag{7.47}$$

From eqs. (7.39), (7.45) and (7.47), we now see that the identity (7.38) holds, verifying the Nielsen identity at one loop. The identity involving the mass, eq. (7.33), can also be verified in a similar manner. However, since this is slightly more involved, we will not go into it here.

7.4 Identities at $T \neq 0$

As we have seen, in chapter 4, the partition function in the presence of gauge interactions is defined in a BRST invariant manner. As a result, the Ward Identities of the theory continue to hold even at finite temperature. Nielsen identities expressing the dependence of the effective potential on the gauge fixing parameter can also be thought of as a sort of Ward Identity and consequently, we expect that these identities should also continue to hold at nonzero temperature. The derivation of the identities goes through without any modification. Verifying the identities is also equally straightforward provided we remember that at finite temperature (in the imaginary time formalism) energy takes discrete values (the description in the real time formalism

is also equally straightforward). Thus, for the gauge field, the σ and the χ fields, we have from eq. (1.29)

$$p^0 \to i\omega_n, \qquad \omega_n = \frac{2n\pi}{\beta} \tag{7.48}$$

The ghost fields, on the other hand, satisfy anti-commutation relations like fermions and we would expect their energy values to be quantized like fermions (see eq. (1.29)). However, as we have discussed in detail in chapter 4 (see the discussion in section 4.6), in spite of their Grassmann nature, the ghost fields do satisfy periodic boundary conditions like any bosonic field so that, for the ghost fields, too, we can write

$$p^0 \to i\omega_n, \qquad \omega_n = \frac{2n\pi}{\beta} \tag{7.49}$$

With these relations (and the discussion on evaluations in the imaginary time formalism in chapter 1), we note that eq.(7.45) takes the form (see also eq. (6.89))

$$\frac{\partial V^1}{\partial \xi} = \frac{1}{2\beta}\sum_n \int \frac{d^3p}{(2\pi)^3} \times \frac{e^2\sigma_c^2(m^2 - \frac{\lambda}{2}\sigma_c^2)}{((\frac{2n\pi}{\beta})^2 + \vec{p}^2 + \xi e^2\sigma_c^2)((\frac{2n\pi}{\beta})^2 + \vec{p}^2 - m^2 + (\frac{\lambda}{2} + \xi e^2)\sigma_c^2)} \tag{7.50}$$

Similarly, eq. (7.47) takes the form

$$C^1 = \frac{e^2\sigma_c}{2\beta}\sum_n \int \frac{d^3p}{(2\pi)^3} \times \frac{1}{((\frac{2n\pi}{\beta})^2 + \vec{p}^2 + \xi e^2\sigma_c^2)((\frac{2n\pi}{\beta})^2 + \vec{p}^2 - m^2 + (\frac{\lambda}{2} + \xi e^2)\sigma_c^2)} \tag{7.51}$$

On the other hand, at finite temperature, the classical Lagrangian density remains the same so that eq. (7.39) remains unchanged. From eqs. (7.39), (7.50) and (7.51), we see that the one loop Nielsen identity, eq. (7.38), continues to hold at finite temperature.

7.5 References

Aitchison, I.J.R and **C.M. Fraser**, Ann. Phys. **156** (1984) 1.

Das, A, unpublished notes.

Dolan, L and **R. Jackiw**, Phys. Rev. **D9** (1974) 2904.

Fukuda, R and **T. Kugo**, Phys. Rev. **D13** (1976) 3469.

Johnston, D.A, Nuc. Phys. **B253** (1985) 687.

Nielsen, N.K, Nuc. Phys. **101** (1975) 173.

Ramaswamy, S, Nuc. Phys. **B453** (1995) 240.

de Wit, B, Phys. Rev. **D12** (1975) 1843.

Chapter 8

Subtleties at $T \neq 0$

Earlier, in chapter 1, we calculated the self-energy diagram for a bosonic field at finite temperature in the imaginary time formalism and showed that this amplitude is nonanalytic at the origin in the energy-momentum space. We can try to calculate the same quantity in the real time formalism and, of course, the answer turns out to be identical. (In fact, we have calculated the imaginary part of the same amplitude, in the real time formalism, using the cutting rules in eq. (5.60) and found that it is identical to the imaginary time calculation.) However, in the process, we learn of subtleties that one should be careful about, in the real time formalism, at finite temperature. In the next section, we will discuss one such subtlety with the example of the degenerate electron gas at zero temperature and will take up a second example in the following section. We could have taken up the

same example of the bosonic self-energy as in chapter 1, but this is a much simpler problem which clarifies the point quite nicely.

Before we go into the discussion, however, a bit of history is necessary to fully appreciate the issue. The self-energy was first calculated in the imaginary time formalism and the nonanalyticity was already noted quite early on. With the popularity of the real time formalism, then, people became interested in checking the imaginary time calculations because such results are unexpected from our experience with calculations at zero temperature. The real time calculations are, however, trickier since they involve products of ill-defined quantities. So, while a nonrigorous calculation, in the real time formalism, showed that the amplitude is nonanalytic as suggested by the imaginary time calculations, there was not sufficient confidence in the manner in which the result was obtained. At the same time, many proponents of real time formalism did not quite accept a nonanalytic behavior and even proposed additional *ad hoc* finite temperature Feynman rules to give the amplitude an analytic structure. To add to the confusion, when a careful, real time calculation was performed with a proper regularization for the ill-defined terms of the propagator, it led to an analytic expression for the self-energy which was in disagreement with the imaginary time result. Therefore, if the two formalisms were to coincide, either the regularization procedure was in error or something else in the calculations needed to be looked at carefully. It turns out that it is something else – in fact, something that we are least likely to suspect, namely, the Feynman combination formula that breaks down (gets modified) at finite temperature.

8.1 Degenerate Electron Gas

Let us consider a system of degenerate electron gas at zero temperature. This is a system that is widely studied and its properties are quite well measured experimentally. Even though we are considering a system at zero temperature, surprisingly enough, it has all the characteristics of a system at finite temperature as we will see shortly. The propagator for a nonrelativistic, degenerate electron gas at zero temperature is well known and has the form

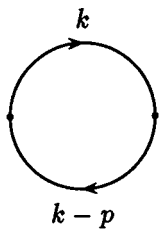
$$\begin{aligned} iS(p) &= \frac{i}{p^0 - \omega_p + \mu + i\epsilon\,\mathrm{sgn}\,(\omega_{\vec{p}} - \mu)} \\ &= \frac{i}{p^0 - \omega_p + \mu + i\epsilon\,\mathrm{sgn}\,(p^0)} \end{aligned} \tag{8.1}$$

Here μ represents the chemical potential which can be identified with the Fermi energy and

$$\omega_p = \frac{\vec{p}^{\,2}}{2m} \tag{8.2}$$

We also note that while the two forms of the propagator in eq. (8.1) are completely equivalent, it is the second form which is calculationally more convenient and, hence, we will use this. Note here that the propagator in eq. (8.1) does not have a well behaved analytic behaviour in the sense that for positive p^0, it has a pole in the lower half plane while for negative values of p^0 the location of the pole is in the upper half of the complex p^0–plane. This is reminiscent of the structure of the propagators at finite temperature (see, for example, eq. (2.37)) even though the system under study is at zero temperature.

We can calculate the self-energy, in the present case, in a direct manner. For simplicity, let us concentrate only on the real part of the self-energy which takes the form



$$\begin{aligned}\mathrm{Re}\,\Pi(p) &= \mathrm{Re}\left((-2i)\int \frac{d^4k}{(2\pi)^4}\, S(k)S(k-p)\right) \\ &= \mathrm{Re}\left((-2i)\int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^0 - \omega_{\vec{k}} + \mu + i\epsilon\mathrm{sgn}\,(k^0)}\right. \\ &\quad \left. \times \frac{1}{k^0 - p^0 - \omega_{\vec{k}-\vec{p}} + \mu + i\epsilon\mathrm{sgn}\,(k^0 - p^0)}\right)\end{aligned} \tag{8.3}$$

where we have normalized the coupling strength to unity and we note that the factor of 2 comes from a trace over the spin index. The propagators can be expanded using the standard representation

$$\lim_{\epsilon\to 0^+} \frac{1}{x + i\alpha\epsilon} = P\frac{1}{x} - i\pi\mathrm{sgn}\,(\alpha)\delta(x) \tag{8.4}$$

Then, eq. (8.3) takes the form

$$\mathrm{Re}\,\Pi(p) = -P\int \frac{d^4k}{(2\pi)^3}$$

$$\times\left(\frac{1}{k^0-\omega_k+\mu}\,\mathrm{sgn}\,(k^0-p^0)\,\delta(k^0-p^0-\omega_{\vec{k}-\vec{p}}+\mu)\right.$$

$$\left.+\frac{1}{k^0-p^0-\omega_{\vec{k}-\vec{p}}+\mu}\,\mathrm{sgn}\,(k^0)\,\delta(k^0-\omega_k+\mu)\right)$$

$$= -P\int \frac{d^4k}{(2\pi)^3}\,\mathrm{sgn}\,(k^0)\delta(k^0-\omega_k+\mu)$$

$$\times\left(\frac{1}{k^0+p^0-\omega_{k+p}+\mu}+(p^\mu\to -p^\mu)\right) \tag{8.5}$$

Here we have let $k^\mu \to k^\mu + p^\mu$ in the first term of the integrand. The k^0 integral can now be simply done, using the delta function, and we obtain

$$\mathrm{Re}\,\Pi(p) =$$

$$-P\int\frac{d^3k}{(2\pi)^3}\,\mathrm{sgn}(\omega_k-\mu)\left(\frac{1}{p^0+\omega_{\vec{k}}-\omega_{k+p}}+(p^\mu\to -p^\mu)\right) \tag{8.6}$$

This is, indeed, the standard expression for the self-energy as can be seen from any traditional text. In fact, if we let

$$\vec{k}\to -\vec{k}$$

in the second term in eq. (8.6), we can write the above expression also as

$$\begin{aligned}
&\mathrm{Re}\,\Pi(p) \\
&= -P\int \frac{d^3k}{(2\pi)^3}\,\mathrm{sgn}\,(\omega_k-\mu)\left(\frac{1}{p^0+\omega_k-\omega_{\vec{k}+\vec{p}}}-\frac{1}{p^0+\omega_{k+p}-\omega_k}\right) \\
&= 2P\int \frac{d^3k}{(2\pi)^3}\theta(\mu-\omega_k)\left(\frac{1}{p^0+\omega_k-\omega_{k+p}}-\frac{1}{p^0+\omega_{\vec{k}+\vec{p}}-\omega_k}\right) \qquad (8.7)
\end{aligned}$$

We have used here the standard representation

$$\mathrm{sgn}(x) = 1 - 2\theta(-x)$$

and the fact that the integrand in the parenthesis is odd under the redefinition

$$\vec{k} \leftrightarrow -(\vec{k}+\vec{p})$$

Furthermore, if we write the chemical potential in terms of a Fermi momentum as

$$\mu = \frac{k_F^2}{2m} \qquad (8.8)$$

and scale all the momentum by the Fermi momentum, namely,

$$\begin{aligned}
\vec{k} &\rightarrow \frac{\vec{k}}{k_F} \\
\vec{p} &\rightarrow \frac{\vec{p}}{k_F} \qquad (8.9)
\end{aligned}$$

as well as define a dimensionless energy as

$$\nu = \frac{p^0}{2\mu} \qquad (8.10)$$

then, the self-energy in eq. (8.7) as

$$\begin{aligned}
&\mathrm{Re}\,\Pi(\nu,p) \\
&= 2mk_F P\int\frac{d^3k}{(2\pi)^3}\theta(1-k)\left(\frac{1}{\nu-\vec{k}\cdot\vec{p}-p^2/2}-\frac{1}{\nu+\vec{k}\cdot\vec{p}+p^2/2}\right) \\
&= 2mk_F P\int\frac{d^3k}{(2\pi)^3}\theta(1-k)\left(\frac{1}{\nu-kp\cos\theta-\frac{p^2}{2}}-\frac{1}{\nu+kp\cos\theta+\frac{p^2}{2}}\right)
\end{aligned} \tag{8.11}$$

Here k and p represent the magnitudes of the three vectors $\vec{k}$ and $\vec{p}$ respectively.

Expression (8.11) is identical to what is known in the standard literature. The integral can be easily evaluated (or directly taken from any standard text) which leads to a self-energy of the form

$$\begin{aligned}
\mathrm{Re}\,\Pi(\nu,p) &= \frac{mk_F}{2\pi^2}\left\{-1+\frac{1}{2p}\left(1-\left(\frac{\nu}{p}-\frac{p}{2}\right)^2\right)\ln\left|\frac{1+(\nu/p-p/2)}{1-(\nu/p-p/2)}\right|\right. \\
&\left.-\frac{1}{2p}\left(1-\left(\frac{\nu}{p}+\frac{p}{2}\right)^2\right)\ln\left|\frac{1+(\nu/p+p/2)}{1-(\nu/p+p/2)}\right|\right\}
\end{aligned} \tag{8.12}$$

This self-energy is nonanalytic at the origin in energy-momentum space and the simplest way to see this is to define

$$\nu=\alpha p \tag{8.13}$$

and take the limit $p\to 0$. In that limit, the self-energy in eq. (8.12) is easily seen to have the form

$$\lim_{p\to 0}\mathrm{Re}\,\Pi(\alpha p,p)=\frac{mk_F}{2\pi^2}\left(-2+\alpha\ln\left|\frac{1+\alpha}{1-\alpha}\right|\right) \tag{8.14}$$

The α dependence shows clearly that the value at the origin depends on how the limits , $p\to 0$ and $\nu\to 0$ are taken.

8.2 Calculation with ϵ–Regularization

In the real time formalism, calculations often involve products of singular functions. The simplest way to see this is to note from eq. (8.5) that if we had set $p^0 = 0$ and $\vec{p} = 0$ in the integrand itself, we would have products of terms like

$$\left(P\frac{1}{x}\right)\delta(x)$$

where the first factor corresponds to the principal value of $\frac{1}{x}$. Such products are highly singular and need to be regularized properly. A standard regularization that is used at finite temperature is known as the ϵ–regularization which simply corresponds to the prescription that we use the representations

$$\begin{aligned} P\frac{1}{x} &= \lim_{\epsilon\to 0^+}\frac{x}{x^2+\epsilon^2} = \lim_{\epsilon\to 0^+}\frac{1}{2}\left(\frac{1}{x+i\epsilon}+\frac{1}{x-i\epsilon}\right) \\ \delta(x) &= \lim_{\epsilon\to 0^+}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2} = \lim_{\epsilon\to 0^+}\frac{i}{2\pi}\left(\frac{1}{x+i\epsilon}-\frac{1}{x-i\epsilon}\right) \end{aligned} \tag{8.15}$$

where it is understood that the limit is to be taken only at the end of the calculation. It is worth checking, in this example of the degenerate electron gas, whether this regularization is, indeed, meaningful or whether it can lead to subtleties. To that end, we will evaluate the self-energy for the degenerate electron gas through this regularization as well.

We note that if we use the representations in eq. (8.15), then we can write the self-energy in eq. (8.5) also as (We will always understand the limiting procedure of (8.15) to be taken at the end although we will not explicitly write it for simplicity.)

$$
\begin{aligned}
\mathrm{Re}\,\Pi(p) &= -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^0+p^0-\omega_{k+p}+\mu+i\epsilon}\right. \\
&\qquad \left. +\frac{1}{k^0+p^0-\omega_{k+p}+\mu-i\epsilon}+(p^\mu\to -p^\mu)\right) \\
&\qquad \times \mathrm{sgn}\,(k^0)\left(\frac{1}{k^0-\omega_k+\mu+i\epsilon}-\frac{1}{k^0-\omega_k+\mu-i\epsilon}\right) \\
&= -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\,\mathrm{sgn}(k^0) \\
&\qquad \left(\frac{1}{k^0+p^0-\omega_{k+p}+\mu+i\epsilon}\,\frac{1}{k^0-\omega_{\vec{k}}+\mu+i\epsilon}\right. \\
&\qquad\quad -\frac{1}{k^0+p^0-\omega_{k+p}+\mu-i\epsilon}\,\frac{1}{k^0-\omega_k+\mu-i\epsilon} \\
&\qquad\quad -\frac{1}{k^0+p^0-\omega_{k+p}+\mu+i\epsilon}\,\frac{1}{k^0-\omega_k+\mu-i\epsilon} \\
&\qquad\quad +\frac{1}{k^0+p^0-\omega_{k+p}+\mu-i\epsilon}\,\frac{1}{k^0-\omega_k+\mu+i\epsilon} \\
&\qquad\quad \left. +(p^\mu\to -p^\mu)\right) \\
&= I_1+I_2 \qquad (8.16)
\end{aligned}
$$

where we have grouped the first two terms with similar analytic behavior of the two factors into I_1 and the other two with opposite analytic behavior of the two factors into I_2.

We note that the integrands in eq. (8.16) for both the terms are nonanalytic functions because of the $\mathrm{sgn}(k^0)$ term. Therefore, to evaluate the k^0 integral, we first rewrite the integrals as follows.

$$
\begin{aligned}
I_1 &= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^4}\int_0^\infty dk^0 \left(\frac{1}{k^0+p^0-\omega_{k+p}+\mu+i\epsilon}\,\frac{1}{k^0-\omega_k+\mu+i\epsilon} \right. \\
&\quad -\frac{1}{k^0-p^0+\omega_{k+p}-\mu-i\epsilon}\,\frac{1}{k^0+\omega_k-\mu-i\epsilon} \\
&\quad -\frac{1}{k^0+p^0-\omega_{k+p}+\mu-i\epsilon}\,\frac{1}{k^0-\omega_k+\mu-i\epsilon} \\
&\quad \left. +\frac{1}{k^0-p^0+\omega_{k+p}-\mu+i\epsilon}\,\frac{1}{k^0+\omega_k-\mu+i\epsilon} + (p^\mu \to -p^\mu) \right)
\end{aligned}
\tag{8.17}
$$

The k^0–integral can now be done in a straightforward manner using the method of residues with a contour in the upper right quadrant and we obtain

$$
\begin{aligned}
I_1 &= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^4}\,2i\pi \left(-\frac{\theta(\mu+p^0-\omega_{k+p})}{p^0+\omega_k-\omega_{k+p}} - \frac{\theta(\mu-\omega_k)}{\omega_{\vec{k}+\vec{p}}-p^0-\omega_k} \right. \\
&\quad \left. -\frac{\theta(\omega_{k+p}-p^0-\mu)}{\omega_{k+p}-p^0-\omega_k} - \frac{\theta(\omega_k-\mu)}{\omega_k-\omega_{k+p}+p^0} + (p^\mu \to -p^\mu) \right) \\
&= -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{p^0+\omega_k-\omega_{k+p}} \left(\mathrm{sgn}(\omega_k-\mu)-\mathrm{sgn}(\omega_{k+p}-p^0-\mu)\right) \right. \\
&\quad \left. + (p^\mu \to -p^\mu) \right)
\end{aligned}
\tag{8.18}
$$

The k^0–integral in I_2 can again be performed exactly in the same manner and yields

$$
I_2 = -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{p^0+\omega_{\vec{k}}-\omega_{k+p}} \left(\mathrm{sgn}(\omega_k-\mu)+\mathrm{sgn}(\omega_{k+p}-p^0-\mu)\right) \right.
$$

$$+ (p^\mu \to -p^\mu)\Bigg) \tag{8.19}$$

From eqs. (8.16), (8.18) and (8.19), then, we obtain

$$\mathrm{Re}\,\Pi(p) = I_1 + I_2 =$$

$$-\int \frac{d^3k}{(2\pi)^3} \mathrm{sgn}(\omega_{\vec{k}} - \mu)\left(\frac{1}{p^0 + \omega_k - \omega_{k+p}} + (p^\mu \to -p^\mu)\right) \tag{8.20}$$

This is, of course, exactly the same result that is obtained by direct integration in eq. (8.6). This shows that the ϵ–regularization does not lead to any particular subtlety if used properly.

8.3 Feynman Parameterization

Next, let us see what would result if we used the naive Feynman combination formula with ϵ–regularization. We note that the naive combination formula gives

$$\frac{1}{A + i\alpha\epsilon}\frac{1}{B + i\beta\epsilon} = \int_0^1 \frac{dx}{[x(A + i\alpha\epsilon) + (1-x)(B + i\beta\epsilon)]^2} \tag{8.21}$$

If we use this formula in eq. (8.17), then we obtain

$$\begin{aligned} I_1 \;=\; -\frac{i}{2}\int \frac{d^3k}{(2\pi)^4}\int_0^\infty\int_0^1 dx \Bigg(& \frac{1}{[k^0 + x(p^0 + \omega_k - \omega_{k+p}) - \omega_k + \mu + i\epsilon]^2} \\ & - \frac{1}{[k^0 - x(p^0 + \omega_k - \omega_{\vec{k}+\vec{p}}) + \omega_k - \mu - i\epsilon]^2} \\ & - \frac{1}{[k^0 + x(p^0 + \omega_k - \omega_{k+p}) - \omega_k + \mu - i\epsilon]^2} \\ & + \frac{1}{[k^0 - x(p^0 + \omega_k - \omega_{k+p}) + \omega_k + i\epsilon]^2} + (p^\mu \to -p^\mu)\Bigg) \end{aligned} \tag{8.22}$$

The integral in eq. (8.22) can be done trivially over k^0 and yields

$$
\begin{aligned}
I_1 &= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^4}\int_0^1 dx \left(\frac{1}{x(p^0+\omega_k-\omega_{k+p})-\omega_k+\mu+i\epsilon}\right. \\
&\qquad +\frac{1}{x(p^0+\omega_k-\omega_{k+p})-\omega_k+\mu+i\epsilon} \\
&\qquad -\frac{1}{x(p^0+\omega_k-\omega_{k+p})-\omega_k+\mu-i\epsilon} \\
&\qquad \left.-\frac{1}{x(p^0+\omega_k-\omega_{\vec{k}+\vec{p}})-\omega_k+\mu-i\epsilon}+(p^\mu\to -p^\mu)\right) \\
&= -i\int \frac{d^3k}{(2\pi)^4}\int_0^1 dx\,(-2i\pi)\left(\delta(x(p^0+\omega_k-\omega_{k+p})-\omega_k+\mu)\right. \\
&\qquad \left.+(p^\mu\to -p^\mu)\right) \\
&= -\int \frac{d^3k}{(2\pi)^3}\int_{-\omega_k+\mu}^{p^0+\mu-\omega_{k+p}} ds\left(\frac{\delta(s)}{p^0+\omega_{\vec{k}}-\omega_{k+p}}+(p^\mu\to -p^\mu)\right) \\
&= -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left[\left\{\Big(\mathrm{sgn}(\omega_{\vec{k}}-\mu)-\mathrm{sgn}(\omega_{k+p}-p^0-\mu)\Big)\right.\right. \\
&\qquad \left.\left.\times\frac{1}{p^0+\omega_{\vec{k}}-\omega_{k+p}}\right\}+(p^\mu\to -p^\mu)\right] \qquad (8.23)
\end{aligned}
$$

Here we have used formula (8.4) in the intermediate steps. We note that this agrees with (8.18) completely.

We can, similarly, evaluate I_2 using the Feynman combination formula. Without going into details, let us simply note here that, in this case, we obtain

$$I_2 = -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left[\left\{2\,\mathrm{sgn}\left(\frac{p^0-\omega_k-\omega_{k+p}}{2}+\mu\right)\right.\right.$$
$$\left.\left.+\,\mathrm{sgn}(\omega_k-\mu)+\mathrm{sgn}(\omega_{k+p}-p^0-\mu)\right\}\right.$$
$$\left.\times\,\frac{1}{p^0+\omega_k-\omega_{k+p}}+(p^\mu\to -p^\mu)\right] \qquad (8.24)$$

This is, however, different from (8.19) and, therefore, it is clear that if we evaluate the self-energy using the naive Feynman combination formula, the result will be inaccurate. It is interesting to note also that while the combination formula in eq. (8.21) seems to work when the factors have the same analyticity properties, it breaks down for factors with opposite analyticity properties. We will next see how the Feynman formula modifies in such cases.

8.4 Modification of Feynman Formula

Let us look at eq. (8.21) and note that if the integrand, on the right hand side, does not have any poles on the real x–axis, then, we have

$$\int_0^1 \frac{dx}{[x(A+i\alpha\epsilon)+(1-x)(B+i\beta\epsilon)]^2}$$
$$= -\frac{1}{A-B+i(\alpha-\beta)\epsilon}\,\frac{1}{x(A-B+i(\alpha-\beta)\epsilon)+B+i\beta\epsilon}\Bigg|_0^1$$
$$= \frac{1}{A+i\alpha\epsilon}\,\frac{1}{B+i\beta\epsilon} \qquad (8.25)$$

This is, of course, the standard combination formula. However, let us assume that the parameters α and β are such that the integrand on the

right hand side of eq. (8.21) has a pole on the real x–axis between 0 and 1. (Note that without loss of generality, we can choose $|\alpha| = |\beta| = 1$.) We see immediately that the integrand can have a pole on the real x–axis at x_0 if

$$x_0(A - B) + B = 0 = (\beta + x_0(\alpha - \beta)) \tag{8.26}$$

The solution to these equations gives

$$\begin{aligned} x_0 &= \frac{\beta}{\beta - \alpha} \\ \beta A &= \alpha B \end{aligned} \tag{8.27}$$

Such a pole will lie between 0 and 1 only if α and β have opposite sign. That is, only when the two factors have opposite analytic behavior will the pole on the real axis contribute nontrivially to the integration. Let us assume that $0 \leq x_0 = \beta/(\beta - \alpha) \leq 1$.

In this case, we can evaluate the right hand side of eq. (8.21) using the principal value. Then, we have

$$\begin{aligned} &\int_0^1 \frac{dx}{[x(A + i\alpha\epsilon) + (1 - x)(B + i\beta\epsilon)]^2} \\ &= \lim_{\eta \to 0^+} \left(\int_0^{\beta/(\beta-\alpha)-\eta} \frac{dx}{[x(A - B + i\epsilon(\alpha - \beta)) + B + i\beta\epsilon]^2} \right. \\ &\quad \left. + \int_{\beta/(\beta-\alpha)+\eta}^1 \frac{dx}{[x(A - B + i\epsilon(\alpha - \beta)) + B + i\beta\epsilon]^2} \right) \\ &= \frac{1}{A + i\alpha\epsilon} \frac{1}{B + i\beta\epsilon} - 2i\pi \frac{(\alpha - \beta)\delta(\beta A - \alpha B)}{A - B + i(\alpha - \beta)\epsilon} \end{aligned} \tag{8.28}$$

In other words, when the two factors have opposite analytic properties, the combination formula (8.21) modifies and, in general, we can write

$$\frac{1}{A+i\alpha\epsilon}\frac{1}{B+i\beta\epsilon} = \int_0^1 \frac{dx}{[x(A+i\alpha\epsilon)+(1-x)(B+i\beta\epsilon)]^2} + 2i\pi\frac{(\alpha-\beta)\delta(\beta A-\alpha B)}{A-B+i(\alpha-\beta)\epsilon} \tag{8.29}$$

Since I_2 involves factors with opposite analyticity, this extra term in the combination formula is likely to contribute there. In fact, we can simply calculate the correction that the δ–function term would give rise to in I_2 and it has the form

$$I_2' = \int \frac{d^3k}{(2\pi)^3}\left[\mathrm{sgn}\left(\frac{p^0 - \omega_k - \omega_{k+p}}{2}+\mu\right)\frac{1}{p^0+\omega_k-\omega_{k+p}} + (p^\mu \to -p^\mu)\right] \tag{8.30}$$

It is now clear that the sum of eqs. (8.24) and (8.30) coincide exactly with (8.19). This simple example shows that since, at finite temperature, the propagator contains terms with opposite analytic behavior, in calculating, one should be careful and use the modified Feynman combination formula whenever necessary. It was, in fact, this particular error that led to contradicting results early on for the self-energy at finite temperature, when calculated in the real time formalism.

8.5 Catalysis at $T \neq 0$

The nonanalyticity at $p^\mu = 0$ for nonzero temperatures is only one of many such nonanalyticities that show up at finite temperature. In this section, we will study in some detail another kind of nonanalyticity which has to do with the vanishing mass limit.

To begin, let us consider a $2+1$ dimensional fermion theory described by

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi \tag{8.31}$$

In $2+1$ dimensions, the spinor representation is 2-dimensional and so, we can think of ψ as a 2-component spinor while we can identify

$$\begin{aligned} \gamma^0 &= \sigma_2 \\ \gamma^1 &= i\sigma_1 \\ \gamma^2 &= -i\sigma_3 \end{aligned} \tag{8.32}$$

Here the three σ_i's represent the three Pauli matrices and

$$\overline{\psi} = \psi^\dagger\gamma^0$$

In this representation, it is clear that there cannot be a γ_5 matrix since

$$\gamma^0\gamma^1\gamma^2 = -iI$$

where I represents the identity matrix and, in fact, there is no other 2×2 matrix which will anti-commute with all the Pauli matrices. As a result, there is no notion of a chiral invariance in this theory.

On the other hand, in $2+1$ dimensions, one can define a parity operation as reflecting through one of the axes, namely,

$$(x,y) \to (-x,y) \quad \text{or,} \quad (x,y) \to (x,-y) \tag{8.33}$$

The reflection about both the axes is equivalent to a rotation by π. Therefore, choosing the first of eq. (8.33) as defining a parity transformation (one can choose the second as well without altering any of the conclusions), we note that, under parity (η is a phase factor corresponding to the intrinsic parity of the field),

$$\psi \to \eta\gamma^1\psi \qquad \overline{\psi} \to -\eta^*\overline{\psi}\gamma^1 \tag{8.34}$$

so that the fermion bilinears transform, under parity, as

$$\begin{aligned} \overline{\psi}\gamma^0\psi &\to \overline{\psi}\gamma^0\psi \\ \overline{\psi}\gamma^1\psi &\to -\overline{\psi}\gamma^1\psi \\ \overline{\psi}\gamma^2\psi &\to \overline{\psi}\gamma^2\psi \end{aligned} \tag{8.35}$$

It is clear now from the definition of eq. (8.34) that, under parity,

$$\overline{\psi}\psi \to -\overline{\psi}\psi \tag{8.36}$$

This shows that the 2-component spinor theory, in $2+1$ dimensions, is not invariant under parity because of the mass term.

On the other hand, we can consider a fermion theory where the fermion field is a 4-component spinor corresponding to a reducible representation. Thus, writing the fermion field in terms of two 2-component spinor fields, ψ_1 and ψ_2 as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{8.37}$$

we see that we can choose the "gamma" matrices to be 4×4 matrices given by

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \\ \gamma^1 &= \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} \\ \gamma^2 &= \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \end{aligned} \tag{8.38}$$

In this reducible representation, it is interesting to note that, we can define two "γ_5" matrices that anti-commute with γ^0, γ^1 and γ^2. Denoting these by

$$\begin{aligned} \gamma^3 &= i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma_5 &= i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{8.39}$$

we recognize that

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Furthermore, it is easy to see that the massive fermion theory of eq. (8.31) (with 4-component fields) is now invariant under the "chiral" transformations

$$\psi \to e^{i\alpha\gamma^3}\psi \qquad \psi \to e^{i\alpha'\gamma_5}\psi \tag{8.40}$$

If $m = 0$, then it is easily checked that the kinetic energy term is invariant under a $U(2)$ transformation where the generators of the transformation are I, γ_5, $-i\gamma^3$ and $\gamma^3\gamma_5$. That is, in such a case, there are two flavors of fermionic fields and $U(2)$ simply represents the flavor symmetry of the massless theory. The mass term, on the other hand, breaks the global flavor symmetry down to the two "chiral" symmetries, *i.e.*, in the presence of the mass term,

$$U(2) \to U(1) \times U(1)$$

The mass, of course, can be thought of as a source for the bilinear $\overline{\psi}\psi$ and, therefore, we see that if in the fermion theory, a condensate develops (when the mass vanishes), then, the flavor symmetry would have broken down to $U(1) \times U(1)$.

It is also interesting to note that, we can now define a parity transformation for the fermion fields as (similar to eq. (8.34))

$$\psi \to -\eta\gamma^1\gamma_5\psi \qquad \overline{\psi} \to \eta^*\overline{\psi}\gamma_5\gamma^1 \tag{8.41}$$

so that, under parity, the bilinears transform as in eq. (8.35). However, it is interesting to note that under parity, eq. (8.41), we now have

$$\overline{\psi}\psi \to \overline{\psi}\psi \tag{8.42}$$

In other words, the massive theory with doubled degrees of freedom is not only invariant under "chiral" transformations, but is also parity invariant.

Let us next study, in some detail, this 4-component theory interacting with a constant external magnetic field along the negative z-axis. First, let us obtain the first quantized spinor wave functions for such a system. The Lagrangian density, in this case, has the form (see eqs. (4.10), (4.11) and (4.18))

$$\mathcal{L} = i\overline{\psi}\gamma^{\mu}(\partial_{\mu} - ieA_{\mu})\psi - m\overline{\psi}\psi \tag{8.43}$$

The equations of motion for the two 2-component spinors can now be obtained and take the form

$$\begin{aligned} (i\sigma_3(\partial_0 - ieA_0) - \sigma_1(\partial_1 - ieA_1) - \sigma_2(\partial_2 - ieA_2) - m)\,\psi_1 &= 0 \\ (i\sigma_3(\partial_0 - ieA_0) - \sigma_1(\partial_1 - ieA_1) - \sigma_2(\partial_2 - ieA_2) + m)\,\psi_2 &= 0 \end{aligned} \tag{8.44}$$

Thus, we see that the form of the two equations satisfied by ψ_1 and ψ_2 are the same except for the sign of the mass term. Therefore, let us analyze only one of them and the other can be obtained from the solutions of this.

For a constant magnetic field along the z-axis, we can choose the vector potentials to be of the form (The choice of the potential is, of course, not unique and can be changed with a gauge transformation, but the physical results would be independent of any such

choice.)

$$A_0 = 0 = A_1 \qquad A_2 = Bx \tag{8.45}$$

where B represents a constant magnetic field. Writing the 2-component spinor field, ψ_1, as

$$\psi_1 = \begin{pmatrix} \psi_1^+ \\ \psi_1^- \end{pmatrix} \tag{8.46}$$

we see that the equation for ψ_1 from eq. (8.44) takes the form

$$\begin{aligned} (i\frac{\partial}{\partial t} - m)\psi_1^+ + (-\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + eBx)\psi_1^- &= 0 \\ (-\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} - eBx)\psi_1^+ - (i\frac{\partial}{\partial t} + m)\psi_1^- &= 0 \end{aligned} \tag{8.47}$$

We note from eq. (8.47) that the solution will have a plane wave form in t and y and, therefore, factoring out

$$e^{-iEt+ipy}$$

the two coupled equations in (8.47) take the form

$$\begin{aligned} (E - m)\psi_1^+ - (\frac{\partial}{\partial \overline{x}} - eB\overline{x})\psi_1^- &= 0 \\ (\frac{\partial}{\partial \overline{x}} + eB\overline{x})\psi_1^+ + (E + m)\psi_1^- &= 0 \end{aligned} \tag{8.48}$$

where we have defined

$$\overline{x} = x - \frac{p}{eB} \tag{8.49}$$

The two first order coupled equations can be decoupled and written in terms of second order equations as

$$\left(\frac{\partial^2}{\partial \overline{x}^2} - e^2B^2\overline{x}^2 + E^2 - m^2 + eB\right)\psi_1^+ = 0$$

$$\left(\frac{\partial^2}{\partial \overline{x}^2} - e^2B^2\overline{x}^2 + E^2 - m^2 - eB\right)\psi_1^- = 0 \qquad (8.50)$$

These are harmonic oscillator equations and, if we assume that $eB > 0$, we can immediately write down the energy eigenvalues by inspection to be

$$E_n^+ = \pm\sqrt{2neB + m^2}$$

$$E_n^- = \pm\sqrt{2(n+1)eB + m^2} \qquad n = 0, 1, 2, \cdots \qquad (8.51)$$

These are basically the Landau levels of the theory and we note that ψ_2 would also have identical energy levels since the energy eigenvalues are invariant under $m \to -m$.

With a little bit of more analysis, we can obtain the normalized eigenfunctions for ψ_1 as well as ψ_2 which are related to the harmonic oscillator solutions. If we now put together both these solutions, then, the 4-component solutions have the form (assuming $m > 0$)

$$\psi_1^+ = N_n e^{-i(|E_n|t - py)} \begin{pmatrix} (|E_n| + m)I(n, p, x) \\ -\sqrt{2eBn}I(n-1, p, x) \\ 0 \\ 0 \end{pmatrix}$$

$$
\begin{aligned}
\psi_2^+ &= N_n e^{-i(|E_n|t-py)} \begin{pmatrix} 0 \\ 0 \\ -\sqrt{2eBn}I(n,p,x) \\ (|E_n|+m)I(n-1,p,x) \end{pmatrix} \\
\psi_1^- &= N_n e^{i(|E_n|t-py)} \begin{pmatrix} \sqrt{2eBn}I(n,-p,x) \\ (|E_n|+m)I(n-1,-p,x) \\ 0 \\ 0 \end{pmatrix} \\
\psi_2^- &= N_n e^{i(|E_n|t-py)} \begin{pmatrix} 0 \\ 0 \\ (|E_n|+m)I(n,-p,x) \\ \sqrt{2eBn}I(n-1,-p,x) \end{pmatrix}
\end{aligned} \tag{8.52}
$$

where we have defined

$$
\begin{aligned}
|E_n| &= \sqrt{2eBn+m^2} \\
N_n &= \frac{1}{\sqrt{2|E_n|(|E_n|+m)}} \\
I(n,p,x) &= \left(\frac{eB}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n[\sqrt{eB}(x-\frac{p}{eB})]e^{-\frac{eB}{2}(x-\frac{p}{eB})^2} \\
I(n=-1,p,x) &= 0
\end{aligned} \tag{8.53}
$$

Here H_n's denote the Hermite polynomials of order n. It is interesting to note from these solutions that, for $n \neq 0$, the energy levels are doubly degenerate whereas for $n = 0$, the two states are nondegenerate and have the forms

$$\psi^+ = e^{-i(|E_0|t-py)} \begin{pmatrix} I(0,p,x) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\psi^- = e^{i(|E_0|t-py)} \begin{pmatrix} 0 \\ 0 \\ I(0,-p,x) \\ 0 \end{pmatrix} \tag{8.54}$$

Let us next note that we can expand the second quantized spinor field operator in the complete basis provided by the solutions in eqs. (8.52)-(8.54) as

$$\psi(\vec{x},t) = \sum_n \sum_{i=1,2}{}' \int \frac{dp}{\sqrt{2\pi}} \left(a_i(n,p)\psi_i^+ + b_i^\dagger(n,p)\psi_i^- \right) \tag{8.55}$$

where the prime stands for the fact that the second sum is only for $n \neq 0$ terms. The fermionic creation and annihilation operators satisfy the standard anti-commutation relations and the nontrivial ones are given by

$$\left[a_i(n,p), a_j^\dagger(n',p')\right]_+ = \delta_{ij}\delta_{nn'}\delta(p-p') = \left[b_i(n,p), b_j^\dagger(n',p')\right]_+ \tag{8.56}$$

Using the field expansion in eq. (8.55) and the anti-commutation relations in eq. (8.56), we can now calculate the value of

the fermion condensate in the presence of a constant, external magnetic field which takes the form

$$\langle 0|\overline{\psi}(\vec{x},t)\psi(\vec{x},t)|0\rangle = -\frac{m}{|m|}\frac{eB}{2\pi} - \frac{meB}{\pi}\sum_{n=1}^{\infty}\frac{1}{|E_n|} \tag{8.57}$$

Here, the second sum is a Hurwitz ζ-function and, as is clear from the form of $|E_n|$ in eq. (8.53), this reduces to the Riemann ζ-function, $\zeta_R(\frac{1}{2})$, when $m \to 0$. The Riemann zeta function, $\zeta_R(s)$ is an analytic function in the entire complex s-plane except for a simple pole at $s = 1$. In fact, for $s = \frac{1}{2}$, the value is known to be $\zeta_R(\frac{1}{2}) = -1.46$ which is finite. Thus, we see that in the limit $m \to 0$, the second term in eq. (8.57) would vanish because of the factor m in the numerator. As a result, we see that

$$\lim_{m\to 0}\langle 0|\overline{\psi}(\vec{x},t)\psi(\vec{x},t)|0\rangle = -\mathrm{sgn}(m)\frac{eB}{2\pi} \tag{8.58}$$

This analysis shows that, in the limit when $m \to 0$, the theory develops a nonzero fermion condensate in the presence of a constant, external magnetic field. As we have discussed earlier, such a nonzero condensate would break the $U(2)$ flavor symmetry of the massless theory down to $U(1) \times U(1)$. A little bit more analysis also would show that even though the value of the fermion condensate is nonzero, it does not imply a mass for the fermions. The phenomenon, nonetheless, is interesting for the following reason. Dynamical mass generation in fermionic theories has been of great interest for a variety of reasons. A particular model which displays such a phenomenon in $3+1$ dimensions is known as the Nambu-Jona-Lasinio model (we will discuss this model in detail in chapter 11) and involves quartic fermion interactions. This

is a nonrenormalizable model in $3+1$ dimensions, but can be thought of as a low energy theory and when the strength of the four-fermion interaction exceeds a critical value, the theory has dynamical generation of mass for the fermions. However, as we have shown above, if, in $2+1$ dimensions, the fermions were to interact with a constant, external magnetic field in addition to the four-fermion interaction, then there would be a nonzero fermion condensate independent of the strength of the four-fermion interaction. This, in turn, would lead to a mass for the fermions (*i.e.*, when the four-fermion terms are shifted around the nonzero condensate, they would give rise to a fermion mass term) for any value of the interaction strength. One says that the presence of an external magnetic field catalyses the dynamical generation of mass in this theory.

All of the above analysis is, of course, at zero temperature and it is interesting to ask what effect would temperature have on such a phenomenon. This question is easily analyzed using the formalism of thermofield dynamics of chapter 3, although other approaches also lead to equivalent results. We recall that in thermofield dynamics, for every field, we have to introduce a tilde field and following eq. (3.30) or (3.70), we define a thermal doublet for the present case as

$$\Psi(\vec{x},t) = \begin{pmatrix} \psi(\vec{x},t) \\ \tilde{\psi}^{\dagger}(\vec{x},t) \end{pmatrix} \tag{8.59}$$

We can now expand the thermal doublet operator in terms of the complete basis of fermionic wave functions as

$$\Psi(\vec{x},t) = \sum_n \sum_{i=1,2}{}' \int \frac{dp}{\sqrt{2\pi}} \left[\begin{pmatrix} a_i(n,p) \\ \tilde{a}_i^\dagger(n,p) \end{pmatrix} \psi_i^+ + \begin{pmatrix} b_i^\dagger(n,p) \\ \tilde{b}_i(n,p) \end{pmatrix} \psi_i^- \right] \tag{8.60}$$

To construct the thermal vacuum and the temperature dependent field operators, we can introduce a thermal Bogoliubov transformation, following the discussions of chapter 3, as

$$U(\theta) = e^{iQ(\theta)} \tag{8.61}$$

where the form of $Q(\theta)$ can be determined (see eq.(3.25)) to be

$$Q(\theta) = i \sum_n \sum_{i=1,2}{}' \int dp \, [\theta_i^+(n)(\tilde{a}_i a_i - a_i^\dagger \tilde{a}_i^\dagger) + \theta_i^-(n)(\tilde{b}_i b_i - b_i^\dagger \tilde{b}_i^\dagger)] \tag{8.62}$$

with

$$\begin{aligned} \sin^2\theta_i^+(n) &= n_F(|E_n|) \\ \sin^2\theta_i^-(n) &= 1 - n_F(-|E_n|) \\ n_F(|E_n|) &= \frac{1}{e^{\beta(|E_n|-\mu)}+1} \end{aligned} \tag{8.63}$$

where we have introduced a chemical potential for generality.

Following the discussion in eqs. (3.27) or (3.41), we can now define the thermal vacuum as

$$|0,\beta\rangle = U(\theta)|0,\tilde{0}\rangle \tag{8.64}$$

Similarly, we can obtain the thermal creation and annihilation operators through the Bogoliubov transformation as

$$\begin{pmatrix} a_i^\beta(n,p) \\ \tilde{a}_i^{\beta\dagger}(n,p) \end{pmatrix} = \begin{pmatrix} \cos\theta_i^+(n) & -\sin\theta_i^+(n) \\ \sin\theta_i^+(n) & \cos\theta_i^+(n) \end{pmatrix} \begin{pmatrix} a_i(n,p) \\ \tilde{a}_i^\dagger(n,p) \end{pmatrix} \tag{8.65}$$

and corresponding ones for b_i and $\tilde{b}_i$ with $\theta_i^+ \to \theta_i^-$.

With all these, we can now calculate the value of the fermion condensate in the thermal vacuum to be

$$\begin{aligned} &\langle 0,\beta|\overline{\psi}(\vec{x},t)\psi(\vec{x},t)|0,\beta\rangle \\ &= \sum_n \sum_{i=1,2}{}' \int \frac{dp}{2\pi} \left(\sin^2\theta_i^+(n)\overline{\psi_i^+}\psi_i^+ + \cos^2\theta_i^-(n)\overline{\psi_i^-}\psi_i^-\right) \\ &= \frac{m}{|m|}\frac{eB}{2\pi}\left(1 - \frac{1}{e^{\beta(|m|-\mu)}+1} - \frac{1}{e^{\beta(|m|+\mu)}+1}\right) \\ &\quad - \frac{meB}{\pi}\sum_{n=1}^{\infty}\frac{1}{|E_n|}\left(1 - \frac{1}{e^{\beta(|E_n|-\mu)}+1} - \frac{1}{e^{\beta(|E_n|+\mu)}+1}\right) \end{aligned} \tag{8.66}$$

It is interesting to note from this equation that for any finite temperature (finite β),

$$\lim_{m\to 0}\langle 0,\beta|\overline{\psi}\psi|0,\beta\rangle = 0 \tag{8.67}$$

In other words, as soon as the system is put in contact with a heat bath, the phase transition breaking the $U(2)$ flavor symmetry disappears. Namely, the dynamical breaking of the symmetry in the presence of an external magnetic field is extremely unstable to the introduction of temperature.

It is more interesting to note from eq. (8.66) that the condensate, in fact, develops a nonanalytic structure with the introduc-

tion of temperature. In fact, let us, for simplicity, set $\mu = 0$ and take the limits

$$m \to 0, \quad \beta \to \infty, \quad |m|\beta = \alpha \tag{8.68}$$

In this limit, the value of the condensate in eq. (8.66) becomes

$$\langle 0, \beta | \overline{\psi}\psi | 0, \beta \rangle = -\mathrm{sgn}(m) \frac{eB}{2\pi} \left(1 - \frac{2}{e^{\alpha} + 1} \right) \tag{8.69}$$

This explicitly shows that the limits $m \to 0$ and $\beta \to \infty$ are not commutative. That is, the value of the condensate is not analytic at $T = 0$. In the (m, T) plane, the value of the condensate is nonanalytic at the origin. This is again similar to the nonanalyticity at the origin in the $(p^0, \vec{p})$ plane that we discussed earlier.

To understand this nonanalyticity a little better, let us again set $\mu = 0$. In this case, we note from eq. (8.63) that

$$\theta_i^+(n) = \theta_i^-(n) \tag{8.70}$$

In fact, there is only one θ in this case satisfying

$$\tan \theta(n) = e^{-\frac{\beta |E_n|}{2}} \tag{8.71}$$

and we can write the generator of the Bogoliubov transformation, in this case, as

$$Q(\theta) = i \sum_n \sum_{i=1,2}{}' \int dp\, \theta(n) (\tilde{a}_i a_i - a_i^\dagger \tilde{a}_i^\dagger + \tilde{b}_i b_i - b_i^\dagger \tilde{b}_i^\dagger) \tag{8.72}$$

For $m \neq 0$, this, indeed, has the right zero temperature behavior, namely, we see from eq. (8.71) that when $\beta \to \infty$,

$$\tan\theta(n) = 0 \quad \Longrightarrow \theta(n) = 0 \tag{8.73}$$

so that $U(\theta) = I$. On the other hand, if we set $m = 0$ and then take the limit $\beta \to \infty$, we note from eq. (8.71) that all the $\theta(n)$'s vanish except

$$\tan\theta(n = 0) = 1 \quad \Longrightarrow \theta(n = 0) = \frac{\pi}{4} \tag{8.74}$$

It is, of course, the ground state that contributes to the condensate and we see that, for this state, θ has the unique value of $\frac{\pi}{4}$. The Bogoliubov transformation, of course, produces a squeezed state which for this particular value of the parameter behaves like a crossed polarizer leading to the vanishing of the condensate in the zero mass limit.

Let us also note that in the limit in eq. (8.68), all the $\theta(n)$'s can be seen to vanish from eq. (8.71) except the one corresponding to the ground state which takes the value

$$\tan\theta(n = 0) = e^{-\frac{\alpha}{2}} \quad \Longrightarrow \theta(n = 0) = \theta(\alpha) \tag{8.75}$$

Namely, we see that the parameter θ itself becomes nonanalytic and, consequently, the generator of the Bogoliubov transformation as well as the transformation itself become nonanalytic at the origin of the (m, T) plane. The Bogoliubov transformation does not necessarily reduce to the Identity operator in the zero temperature limit – rather, the structure depends on how this limit is taken. It is clear now that, because the Bogoliubov transformation develops a nonanalytic structure, not only the condensate, but most other observables are also likely to inherit this nonanalytic structure.

8.6 References

Appelquist, T et al, Phys. Rev. **D33** (1986) 3704.

Bedaque, P.F and **A. Das**, Phys. Rev. **D45** (1992) 2906.

Bedaque, P.F and **A. Das**, Phys. Rev. **D47** (1993) 601.

Das, A and **M. Hott**, Phys. Rev. **D53** (1996) 2252.

Fujimoto, Y and **H. Yamada**, Z. für Phys. **C37** (1988) 265.

Gribosky, P.S and **B.R. Holstein**, Z. für Phys. **C47** (1990) 205.

Gusynin, V.P et al, Phys. Rev. Lett. **73** (1994) 3499.

Klimenko, K.G, Z. Phys. **C54** (1992) 323.

Weldon, H, Phys. Rev. **D47** (1993) 594.

Chapter 9

Supersymmetry at $T \neq 0$

9.1 Supersymmetry

As we have seen in chapter 6, most symmetries which are spontaneously broken at zero temperature are restored at some finite critical temperature. That is, field theories with spontaneously broken symmetries, behave like magnets where the magnetization vanishes beyond the Curie temperature restoring rotational invariance. There are, of course, some exceptions and one such case that we discussed earlier is a system with a dynamically broken symmetry. There is another fundamental symmetry, known as supersymmetry, that is not restored at finite temperature either – in fact, it is spontaneously broken at finite temperature. Since supersymmetry is quite fundamental in the study

of various grand unified theories as well as in the study of various cosmological questions, we will discuss supersymmetry and its behavior at finite temperature in some detail in this chapter.

Most ordinary symmetries are based on Lie Groups and the multiplets of these symmetry groups consist of particles of the same kind, namely, either bosons or fermions. Therefore, most ordinary symmetry transformations map either bosons into bosons or fermions into fermions. Supersymmetry, on the other hand, is a symmetry which transforms bosons into fermions and *vice versa*. This is the ultimate form of symmetry that one can imagine in the sense that it would imply that bosons and fermions are different manifestations of a single entity.

Since supersymmetry transformations take bosons into fermions and *vice versa*, it is clear that, unlike ordinary symmetries, the generators of supersymmetry will be fermionic in character. The algebra of these generators, which would involve anti-commutators, would clearly give rise to a generator which would have a bosonic character and, consequently, would satisfy commutation relations. As a result, we see that the complete algebra of supersymmetry would correspond to a graded algebra which would involve both commutators and anti-commutators. To see the simplest of such algebras, let us assume that Q and $\overline{Q}$ are two fermionic conserved charges in a quantum mechanical theory described by a quantum mechanical Hamiltonian, H. Then, the simplest graded algebra that we can write down, involving these three generators, has the form

$$\begin{aligned} [H,H] &= 0 \\ [Q,Q]_+ &= 0 = \left[\overline{Q},\overline{Q}\right]_+ \\ \left[Q,\overline{Q}\right]_+ &= H \end{aligned} \tag{9.1}$$

As we will see, this is the algebra satisfied by the conserved charges in the supersymmetric, nonrelativistic quantum mechanics. We can also easily check in this simple algebra that the super-Jacobi identity defined by

$$(-1)^{|A||C|}\left[\left[A,B\right\},C\right\}+(-1)^{|B||C|}\left[\left[C,A\right\},B\right\}+(-1)^{|A||B|}\left[\left[B,C\right\},A\right\}=0 \tag{9.2}$$

is satisfied. Here $|\phi|$ represents the Grassmann parity of any variable ϕ (which takes the value 0 for bosons and 1 for fermions) and the bracket for any two variables is defined to be

$$[A,B\} = AB - (-1)^{|A||B|}BA \tag{9.3}$$

The relativistic supersymmetric field theories, on the other hand, are realizations of the graded Poincare algebra. Let P_μ and $M_{\mu\nu}$ represent the usual translation and Lorentz generators of the Poincare algebra. In addition, if we have two fermionic generators represented by the Majorana spinors, Q_α and $\overline{Q}_\alpha$, $\alpha = 1,2,3,4$, then, the graded Poincare algebra has the additional relations (*i.e.*, in addition to the usual Poincare algebra relations)

$$
\begin{aligned}
[P_\mu, Q_\alpha] &= 0 = \left[P_\mu, \overline{Q}_\alpha\right] \\
[M_{\mu\nu}, Q_\alpha] &= -\frac{1}{2}(\sigma_{\mu\nu})_{\alpha\beta} Q_\beta \\
\left[M_{\mu\nu}, \overline{Q}_\alpha\right] &= -\frac{1}{2}(\sigma_{\mu\nu})_{\alpha\beta} \overline{Q}_\beta \\
\left[Q_\alpha, \overline{Q}_\beta\right]_+ &= (\gamma^\mu)_{\alpha\beta} P_\mu
\end{aligned}
\tag{9.4}
$$

where γ^μ's represent the Dirac matrices and

$$
\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right] \tag{9.5}
$$

The relations, in eq. (9.4), can be easily understood as follows. The first relation merely says that supersymmetry transformations commute with translations so that supersymmetric theories will be translation invariant. The second and the third give the transformation properties of Q_α and $\overline{Q}_\alpha$ under Lorentz transformations, namely, they transform as spinors. The last, on the other hand, represents the fact that two supersymmetry transformations are equivalent to a translation. (One can also assume the super charges, Q_α and $\overline{Q}_\alpha$, carry an internal symmetry index which would, then, lead to an extended graded algebra. However, we will not go into this.) The representations of the supersymmetry algebra can be easily worked out much the same way as is done for ordinary Lie algebras and consist of multiplets containing bosons and fermions of degenerate mass (or energy in the non-relativistic case). It is in this sense that supersymmetry unifies the two different kinds of particles known in nature. The supersymmetry

generators acting on a bosonic (fermionic) state take it to a fermionic (bosonic) state of same mass (energy). This is easily seen from eq. (9.4) as follows (it can also be seen from eq. (9.1)). Let $|\psi\rangle$ represent a state with energy-momentum p^μ. Thus,

$$\begin{aligned} P^\mu|\psi\rangle &= p^\mu|\psi\rangle \\ P^\mu(Q_\alpha|\psi\rangle) &= Q_\alpha P^\mu|\psi\rangle = p^\mu(Q_\alpha|\psi\rangle) \end{aligned} \tag{9.6}$$

The equality of masses for the superpartner states, $|\psi\rangle$ and $(Q_\alpha|\psi\rangle)$, then, follows from $p^2 = p^\mu p_\mu = m^2$. (An alternate way of seeing this is to note that the only Casimir operator of the graded Poincare algebra is $P^2 = P^\mu P_\mu$ and, hence, all representations must consist of multiplets with degenerate mass.)

The supersymmetric theories are of interest for various reasons. The supersymmetric field theories have extremely well behaved ultraviolet properties. The bosonic and fermionic loop contributions in many physical processes cancel leading to a much milder divergence structure than in ordinary field theories. Supersymmetry also gives a physical reason for the existence of scalar fields – they can simply be thought of as the superpartners of fermionic fields. Furthermore, supersymmetry implies a vanishing ground state energy which is quite significant in connection with the question of the cosmological constant. The vanishing of the ground state energy can be easily seen from eq. (9.1) (or equally well from eq. (9.4)) as follows. If $|0\rangle$ represents the ground state, then, from eq. (9.1), we have

$$\langle 0|Q\overline{Q} + \overline{Q}Q|0\rangle = \langle 0|H|0\rangle = E_0 \tag{9.7}$$

If supersymmetry is unbroken, then, we have

$$Q|0\rangle = 0 = \overline{Q}|0\rangle \tag{9.8}$$

It follows from this, as well as from eq. (9.7), that, if supersymmetry is a true symmetry of the theory, then the ground state energy must vanish, namely,

$$E_0 = 0 \tag{9.9}$$

Incidentally, we can generalize the argument in the above equation to show that the energy of any state in a supersymmetric theory is necessarily positive semi-definite.

While supersymmetry is a symmetry with a rich structure, experimentally, supersymmetry is unobserved. In fact, the spectrum of known elementary particles does not display any degeneracy in the masses of bosons and fermions. Therefore, it is clear that if supersymmetry is a symmetry of our physical theory, it must be spontaneously broken so as to maintain all the good features following from the symmetry while avoiding degenerate bosonic and fermionic states. Furthermore, we are interested in the question of whether thermal effects lead to a spontaneous breakdown of supersymmetry. Intuitively, it is clear that since bosons and fermions obey different statistics, any symmetry that transforms bosons into fermions and *vice versa* would be broken at finite temperature. However, we would like to study the structure of this systematically. To study the phenomenon of supersymmetry breaking, we again have to look for order parameters associated with this symmetry. It is true that when supersymmetry is spontaneously broken, as in ordinary symmetries, the charges do not annihilate the

vacuum

$$\begin{aligned} Q|0\rangle &\neq 0 \\ \overline{Q}|0\rangle &\neq 0 \end{aligned} \tag{9.10}$$

This, also implies, as in the case of spontaneous breaking of an ordinary symmetry (see, for example, eq. (6.17)), that the expectation value of the transformation of a dynamical variable under supersymmetry must be nonzero. The only difference is that in this case, the dynamical variable whose transformation picks up a nonzero value must be fermionic. However, it may not always be easy to check whether the charges annihilate the ground state or not. On the other hand, if the charges do not annihilate the vacuum, then, it follows, from eqs. (9.7) and (9.10) that when supersymmetry is spontaneously broken, the ground state energy will be nonzero (and positive), namely,

$$E_0 > 0 \tag{9.11}$$

Thus, we can think of the ground state energy in a supersymmetric theory as an order parameter of supersymmetry breaking.

There is also an alternate order parameter of supersymmetry breaking known as the Witten index. Let us define

$$(-1)^{N_F} = e^{i\pi N_F} = 1 - 2N_F \tag{9.12}$$

where N_F represents the fermion number operator and takes the values $0, 1$ in any state. For the bosonic states, the fermion number takes the value 0 while the value 1 holds for fermionic states. Clearly, therefore, if we define the Witten index as

$$\Delta = \langle 0|(-1)^{N_F}|0\rangle = \sum_n \langle n|(-1)^{N_F}|n\rangle = \mathrm{Tr}(-1)^{N_F} \tag{9.13}$$

then, this measures the difference in the number of zero energy bosonic and fermionic states in the theory. The second equality follows from the fact that all the higher energy bosonic and fermionic states are paired giving zero to the sum. If the Witten index in eq. (9.13) takes a nonzero value, then, one can show that supersymmetry is unbroken. In fact, it is clear that the ground state, when supersymmetry is unbroken, is bosonic and has no superpartner state giving a value 1 to the index. However, when the value of the index vanishes, supersymmetry may be spontaneously broken. For example, if the number of zero energy bosonic as well as fermionic states vanishes, then, the Witten index will identically vanish. In this case, however, there will be no zero energy state and, consequently, it follows from the supersymmetry algebra that supersymmetry must be spontaneously broken. Thus, the Witten index can also be taken as an order parameter in the study of spontaneous breakdown of supersymmetry.

In the next section, we will study some simple quantum mechanical systems to bring out the various features in connection with supersymmetry breaking.

9.2 Supersymmetric Oscillator

Let us begin with the simplest of supersymmetric theories in this section. The Lagrangian for a general one-dimensional supersymmetric quantum mechanical theory is given by

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}(f(q))^2 + i\overline{\psi}\dot{\psi} - \frac{1}{2}f'(q)\left[\overline{\psi}, \psi\right] \tag{9.14}$$

where $f(q)$ is an arbitrary polynomial of the coordinate q and may involve various coupling constants and other parameters of the theory. ψ and $\overline{\psi}$ are treated as completely independent anti-commuting quantities describing the fermionic degrees of freedom of the theory. (Supersymmetric theories normally involve some auxiliary fields for the closure of the algebra. In writing the Lagrangian completely in terms of dynamical variables as in eq. (9.14), we have eliminated the auxiliary fields. Furthermore, for certain classes of $f(q)$, it is known that instanton solutions are present leading to a breaking of supersymmetry. In all our discussions, we will assume that $f(q)$ has the form such that instantons are not present.) It can be easily checked that the action obtained from the Lagrangian in eq. (9.14) is invariant under the following two sets of supersymmetry transformations

$$\begin{aligned}
&\delta q = \tfrac{1}{\sqrt{2}}\overline{\psi}\epsilon && \overline{\delta} q = \tfrac{1}{\sqrt{2}}\overline{\epsilon}\psi \\
&\delta\psi = -\tfrac{i}{\sqrt{2}}\dot{q}\epsilon - \tfrac{1}{\sqrt{2}}f(q)\epsilon && \overline{\delta}\psi = 0 \\
&\delta\overline{\psi} = 0 && \overline{\delta}\,\overline{\psi} = \tfrac{i}{\sqrt{2}}\dot{q}\overline{\epsilon} - \tfrac{1}{\sqrt{2}}f(q)\overline{\epsilon}
\end{aligned} \tag{9.15}$$

Here ϵ and $\overline{\epsilon}$ are two constant anti-commuting parameters of the transformations and the structure of the transformations in eq. (9.15) shows clearly that under a supersymmetry transformation, $q \leftrightarrow \psi\,(\overline{\psi})$, namely, the bosonic and the fermionic degrees of freedom are transformed into each other.

The canonical momenta can be obtained from the Lagrangian in eq. (9.14) and the Hamiltonian can be constructed to be

$$H = \frac{1}{2}\left[p^2 + (f(q))^2 + f'(q)\left[\overline{\psi}, \psi\right]\right] \tag{9.16}$$

and the fundamental nontrivial commutation and anti-commutation relations of the theory are given by ($\hbar = 1$)

$$[q, p] = i \qquad \left[\psi, \overline{\psi}\right]_+ = 1 \tag{9.17}$$

The supersymmetry charges can now be constructed to be

$$Q = \frac{1}{\sqrt{2}}(p + if(q))\overline{\psi} \qquad \overline{Q} = \frac{1}{\sqrt{2}}(p - if(q))\psi \tag{9.18}$$

and it can be easily checked with the (anti) commutation relations of eq. (9.17) that these charges, indeed, generate the supersymmetry transformations of eq. (9.15). Furthermore, Q, $\overline{Q}$ and H can also be easily seen through the (anti) commutation relations of eq. (9.17) to satisfy the supersymmetry algebra of eq. (9.1).

Let us next specialize to the case where $f(q)$ is a linear function of the bosonic coordinate q,

$$f(q) = \omega q \tag{9.19}$$

In this case, the Hamiltonian of the system becomes

$$H = \frac{1}{2}\left[p^2 + \omega^2 q^2 + \omega\left[\overline{\psi}, \psi\right]\right] \tag{9.20}$$

and corresponds to the Hamiltonian for the supersymmetric oscillator (with $m = 1$). To see this, let us define, as usual, the bosonic and the fermionic creation and annihilation operators as

$$\begin{aligned} a_B &= \tfrac{1}{\sqrt{2\omega}}(p - i\omega q) \quad & a_B^\dagger &= \tfrac{1}{\sqrt{2\omega}}(p + i\omega q) \\ a_F &= \overline{\psi} & a_F^\dagger &= \psi \end{aligned} \tag{9.21}$$

and we can derive the nontrivial canonical (anti) commutation relations between these operators to be

$$\left[a_B, a_B^\dagger\right] = 1 = \left[a_F, a_F^\dagger\right]_+ \tag{9.22}$$

The Hamiltonian of eq. (9.20) can then be written in terms of the creation and the annihilation operators as

$$H = \omega(a_B^\dagger a_B + a_F^\dagger a_F) \tag{9.23}$$

which is, indeed, the Hamiltonian for the supersymmetric oscillator. The supersymmetry charges, in this case, take the form (see eqs. (9.18), (9.19) and (9.21))

$$Q = \sqrt{\omega}\, a_B^\dagger a_F \qquad \overline{Q} = \sqrt{\omega}\, a_F^\dagger a_B \tag{9.24}$$

Once again, we can check using the (anti) commutation relations of eq. (9.22) that Q, $\overline{Q}$ and H of eqs. (9.24) and (9.23) satisfy the supersymmetry algebra of eq. (9.1).

The Hilbert space of the supersymmetric oscillator can be worked out in a straightforward manner. Let us define the bosonic and the fermionic number operator as usual as

$$N_B = a_B^\dagger a_B \qquad N_F = a_F^\dagger a_F \tag{9.25}$$

so that the Hamiltonian of eq. (9.23) can also be written as

$$H = \omega(N_B + N_F) \tag{9.26}$$

The eigenvalues of the number operators can be determined in the standard manner to take the values

$$n_B = 0, 1, 2, \cdots \qquad n_F = 0, 1 \tag{9.27}$$

The eigenstates of the Hamiltonian, $|n_B, n_F\rangle$, can now be seen to satisfy

$$H|n_B, n_F\rangle = \omega(n_B + n_F)|n_B, n_F\rangle \tag{9.28}$$

with n_B and n_F taking the values of eq. (9.27). The ground state of the theory, as usual, satisfies

$$a_B|0\rangle = 0 = a_F|0\rangle \tag{9.29}$$

The higher energy states are, then, explicitly determined to have the form

$$|n_B, n_F\rangle = \frac{(a_B^\dagger)^{n_B}}{\sqrt{n_B!}}(a_F^\dagger)^{n_F}|0\rangle \tag{9.30}$$

Let us now note from eqs. (9.27) and (9.28) that, for any state $|n_B, n_F\rangle$,

$$\langle n_B, n_F|H|n_B, n_F\rangle = \omega(n_B + n_F) \geq 0 \tag{9.31}$$

as it should be for a supersymmetric theory. In particular, we note that

$$\langle 0|H|0\rangle = 0 \tag{9.32}$$

That is, the ground state energy, unlike the case of a bosonic or a fermionic oscillator, vanishes. This would further emphasize that supersymmetry is an unbroken symmetry in this theory. In fact, we can check explicitly from the definition of eq. (9.24) as well as the ground state condition of eq. (9.29) that

$$Q|0\rangle = 0 = \overline{Q}|0\rangle \tag{9.33}$$

We note from eqs. (9.27) and (9.28) that, except for the ground state, all the energy eigenstates are doubly degenerate. In fact, we see that the effect of the supercharges acting on the energy eigenstates are given by (for $n_B \neq 0$)

$$\begin{aligned} Q|n_B,1\rangle &= \sqrt{\omega(n_B+1)}|n_B+1,0\rangle \\ Q|n_B,0\rangle &= 0 \\ \overline{Q}|n_B,0\rangle &= \sqrt{\frac{\omega}{n_B}}|n_B-1,1\rangle \\ \overline{Q}|n_B,1\rangle &= 0 \end{aligned} \tag{9.34}$$

It is now clear that the superpartner states $|n_B,0\rangle$ and $|n_B-1,1\rangle$ (for $n_B \neq 0$) are degenerate in energy as we would expect in a supersymmetric theory. The Witten index can also be calculated for this theory and has the form

$$\Delta = \langle 0|(1-2N_F)|0\rangle = 1 = \mathrm{Tr}(1-2N_F) \tag{9.35}$$

Thus, we see that all the order parameters, in the case of the supersymmetric oscillator, indicate that supersymmetry is a true symmetry of this theory at zero temperature.

Let us next analyze this model at finite temperature. Since we are interested in the structure of the Hilbert space and the action of various operators on the space of states, in this case, it is more convenient to use the formalism of thermofield dynamics discussed in chapter 3. As we have seen there, in this formalism one doubles the degrees of freedom by introducing "tilde" variables (see, for example, eqs. (3.16) and (3.35)). The combined Hamiltonian which governs the dynamics of the "tilde" and the "non-tilde" systems, in this case, can be written as

$$\begin{aligned} \hat{H} &= H - \tilde{H} \\ &= m(a_B^\dagger a_B + a_F^\dagger a_F) - m(\tilde{a}_B^\dagger \tilde{a}_B + \tilde{a}_F^\dagger \tilde{a}_F) \end{aligned} \tag{9.36}$$

We assume, as was discussed in chapter 3, that the "tilde" operators satisfy the same commutation (anti-commutation) relations as the original operators and that the "tilde" and the "non-tilde" operators commute (anti-commute) among each other.

The states, in this doubled Hilbert space can be labelled as

$$\begin{aligned} |0\rangle &= |0\rangle \otimes |\tilde{0}\rangle \\ |n_B, n_F; \tilde{n}_B, \tilde{n}_F\rangle &= |n_B, n_F\rangle \otimes |\tilde{n}_B, \tilde{n}_F\rangle \end{aligned} \tag{9.37}$$

We can now construct the thermal vacuum following the discussion of chapter 3 (see eqs. (3.25)-(3.27) as well as (3.41)) as

$$|0, \beta\rangle = U(\theta)|0\rangle = e^{-iG(\theta)}|0\rangle \tag{9.38}$$

where the generator of the Bogoliubov transformation is given by (see eqs. (3.28) as well as (3.42))

$$\begin{aligned} G(\theta) &= -i\theta_F(\beta)(\tilde{a}_F a_F - a_F^\dagger \tilde{a}_F^\dagger) - i\theta_B(\beta)(\tilde{a}_B a_B - a_B^\dagger \tilde{a}_B^\dagger) \\ \tan\theta_F(\beta) &= e^{-\beta\omega/2} = \tanh\theta_B(\beta) \end{aligned} \tag{9.39}$$

(Note that we are denoting, in this chapter, the generator of the Bogoliubov transformation by $G(\theta)$, which is different from that in chapter 3, simply because conventionally, the supersymmetric charges are represented by Q.)

Once the Bogoliubov transformation defining the thermal vacuum has been determined, we can obtain the thermal operators corresponding to any given zero temperature operator A simply as

$$A(\beta) = U(\theta) A U^\dagger(\theta)$$

In particular, this determines the thermal creation and annihilation operators as the linear combinations

$$\begin{aligned} a_B(\beta) &= a_B \cosh\theta_B - \tilde{a}_B^\dagger \sinh\theta_B \\ \tilde{a}_B(\beta) &= \tilde{a}_B \cosh\theta_B - a_B^\dagger \sinh\theta_B \\ a_F(\beta) &= a_F \cos\theta_F - \tilde{a}_F^\dagger \sin\theta_F \\ \tilde{a}_F(\beta) &= \tilde{a}_F \cos\theta_F + a_F^\dagger \sin\theta_F \end{aligned} \tag{9.40}$$

and their hermitian conjugates. Of course, these relations can be inverted and the zero temperature operators can also be expressed in terms of the thermal creation and annihilation operators as

$$
\begin{aligned}
a_B &= a_B(\beta)\cosh\theta_B - \tilde{a}_B^\dagger(\beta)\sinh\theta_B \\
\tilde{a}_B &= \tilde{a}_B(\beta)\cosh\theta_B - a_B^\dagger(\beta)\sinh\theta_B \\
a_F &= a_F(\beta)\cos\theta_F + \tilde{a}_F^\dagger(\beta)\sin\theta_F \\
\tilde{a}_F &= \tilde{a}_F(\beta)\cos\theta_F - a_F^\dagger(\beta)\sin\theta_F
\end{aligned} \tag{9.41}
$$

The thermal operators, of course, satisfy the same commutation (anti-commutation) relations as the zero temperature ones as can be explicitly checked. Furthermore, the thermal Hilbert space can be constructed from the thermal vacuum by the operation of the thermal creation operators. We note, in particular, that the thermal vacuum is normalized to unity since the Bogoliubov transformation is a unitary transformation (in general, at least formally). Furthermore, the thermal vacuum is annihilated by all the thermal annihilation operators, namely, for $a(\beta) = a_B(\beta), \tilde{a}_B(\beta), a_F(\beta), \tilde{a}_F(\beta)$,

$$
a(\beta)|0,\beta\rangle = 0 \tag{9.42}
$$

We are now ready to calculate various statistical averages of interest in this theory. First, we note that the energy of the thermal vacuum is given by

$$
\begin{aligned}
E_0(\beta) &= \langle 0,\beta|H|0,\beta\rangle = \langle 0,\beta|\omega(a_B^\dagger a_B + a_F^\dagger a_F)|0,\beta\rangle \\
&= \omega(\sinh^2\theta_B + \sin^2\theta_F) \\
&= \omega\left(\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} + \frac{e^{-\beta\omega}}{1+e^{-\beta\omega}}\right)
\end{aligned} \tag{9.43}
$$

where we have used eqs. (9.41) and (9.42) as well as the commutation (anti-commutation) relations of the thermal creation and annihilation operators.

There are several things to note from eq. (9.43). First, for $T \to 0$ or equivalently, $\beta \to \infty$, $E_0 \to 0$ as we would expect at zero temperature. At any finite temperature, however, the thermal energy is nonzero and positive. In fact, it is simply the sum of the thermal energies for a bosonic and a fermionic oscillator system of the same frequency. This is the first indication that supersymmetry may be spontaneously broken at finite temperature. In fact, since there is no state with zero energy, from the discussion following eq. (9.13), we expect the Witten index to vanish corresponding to the supersymmetry broken phase. However, we also point out that in this theory, the value of the Witten index can be explicitly calculated and leads to the following result

$$\begin{aligned} \Delta(\beta) &= \langle 0,\beta|(1-2N_F)|0,\beta\rangle = \langle 0,\beta|(1-2a_F^\dagger a_F)|0,\beta\rangle \\ &= (1-2\sin^2\theta_F) \\ &= \frac{1-e^{-\beta\omega}}{1+e^{-\beta\omega}} \end{aligned} \tag{9.44}$$

We note here that for $T \to 0$ ($\beta \to \infty$), $\Delta(\beta) \to 1$ consistent with eq. (9.35). Furthermore, for $T \to \infty$ ($\beta \to 0$), $\Delta(\beta) \to 0$ consistent with the fact that supersymmetry is spontaneously broken at very high temperatures. At any finite temperature, however, eq. (9.44) gives a fractional value for the Witten index. This shows that at finite temperatures, $(1-2N_F)$ does not give a good representation for the Witten index and one should calculate this quantity more carefully. (By defi-

nition, eq. (9.13), Witten index measures the difference in the number of zero energy bosonic and fermionic states and, therefore, must be an integer.)

In this simple model, we can, of course, explicitly check whether supersymmetry is spontaneously broken from the action of the supersymmetry charges on the thermal vacuum. In fact, we note from eq. (9.24) that

$$\begin{aligned}
Q|0,\beta\rangle &= \sqrt{\omega} a_B^\dagger a_F |0,\beta\rangle \\
&= \sqrt{\omega}\cosh\theta_B \sin\theta_F \\
&\times \; |n_B(\beta)=1, n_F(\beta)=0; \tilde{n}_B(\beta)=0, \tilde{n}_F(\beta)=1\rangle \\
&= \frac{\sqrt{\omega} e^{-\beta\omega/2}}{[(1-e^{-\beta\omega})(1+e^{-\beta\omega})]^{1/2}} |\chi_1(\beta)\rangle \\
\overline{Q}|0,\beta\rangle &= \sqrt{\omega} a_F^\dagger a_B |0,\beta\rangle \\
&= \sqrt{\omega}\sinh\theta_B \cos\theta_F \\
&\times \; |n_B(\beta)=0, n_F(\beta)=1; \tilde{n}_B(\beta)=1, \tilde{n}_F(\beta)=0\rangle \\
&= \frac{\sqrt{\omega} e^{-\beta\omega/2}}{[(1-e^{-\beta\omega})(1+e^{-\beta\omega})]^{1/2}} |\chi_2(\beta)\rangle
\end{aligned} \tag{9.45}$$

This shows explicitly that as $T \to 0$ ($\beta \to \infty$), the supercharges annihilate the vacuum consistent with our earlier discussion that at zero temperature supersymmetry is unbroken. However, at any finite temperature, we see that the supercharges do not annihilate the thermal vacuum indicating that supersymmetry is spontaneously broken at finite temperature. Intuitively, it is clear that since supersymmetry transforms

bosons into fermions and since bosons and fermions respond differently to temperature (their statistics are quite distinct), supersymmetry will be broken with the introduction of a heat bath or for that matter any boundary condition which will distinguish between the two. What is interesting, however, is the nature of the breaking, namely, the fact that supersymmetry is spontaneously broken. In fact, following our discussion in chapter 6, we recognize that the states $|\chi_1(\beta)\rangle$ and $|\chi_2(\beta)\rangle$ represent the Goldstino states at finite temperature. (The Goldstone states, in this case, are fermionic since the generators of symmetry, Q and $\overline{Q}$ are and, conventionally, such states are known as Goldstino states.)

9.3 Interacting Theory

The supersymmetric oscillator is a free theory. Therefore, while the discussion of the last section clearly shows that supersymmetry is spontaneously broken in this theory at finite temperature, a question naturally arises as to whether interactions can change this result. More specifically, it will be nice to know whether in an interacting theory, the ground state energy would continue to be nonzero at finite temperature. This question is particularly significant because, as we know, in a supersymmetric theory the bosonic and fermionic contributions tend to cancel leading to a theory which is much better behaved than theories without supersymmetry. Intuitively, however, we know that we are looking for thermal contributions and they generally have a tendency to add rather than cancel. It is, therefore, of interest to analyze this question in some detail.

Let us, for simplicity, consider a one-dimensional, interacting supersymmetric quantum mechanical theory described by the Lagrangian (see eq. (9.14))

$$L = \frac{1}{2}(\dot{q})^2 - \frac{1}{2}(\xi + \omega q + g q^2)^2 + i\overline{\psi}\dot{\psi} - \frac{1}{2}(\omega + 2gq)\left[\overline{\psi}, \psi\right] \qquad (9.46)$$

Here ξ is a constant parameter known as the Fayet-Illiopoulous parameter which can lead to supersymmetry breaking in more realistic field theories. For $\xi > 0$, supersymmetry is normally spontaneously broken in realistic field theories while when $\xi < 0$, supersymmetry is unbroken. The sign of this term, therefore, is quite crucial in realistic field theories to determine whether supersymmetry is spontaneously broken. Furthermore, g represents the strength of interaction which controls both the Yukawa as well as the quartic interactions which is a standard feature of supersymmetric theories.

We can now obtain the Hamiltonian for the interacting theory in eq. (9.46) in a straightforward manner which takes the form

$$\begin{aligned} H &= \frac{p^2}{2} + \frac{1}{2}(\xi + \omega q + g q^2)^2 + \frac{1}{2}(\omega + 2gq)\left[\overline{\psi}, \psi\right] \\ &= \frac{p^2}{2} + \frac{\omega^2 q^2}{2} + \frac{\omega}{2}\left[\overline{\psi}, \psi\right] \\ &\quad + \frac{1}{2}[(\xi + g q^2)^2 + 2\omega q(\xi + g q^2)] + gq\left[\overline{\psi}, \psi\right] \\ &= H_0 + H_I \end{aligned} \qquad (9.47)$$

Here H_0 is the Hamiltonian for the supersymmetric oscillator (see eq. (9.20)) which we have already studied in the last section and H_I represents all the interactions and has the form

$$H_I = \frac{1}{2}[(\xi + gq^2)^2 + 2\omega q(\xi + gq^2)] + gq\left[\overline{\psi}, \psi\right] \tag{9.48}$$

The interaction Hamiltonian, (9.48), can be rewritten in terms of the creation and the annihilation operators with the identifications in the last section (see eq. (9.21)) and takes the form

$$\begin{aligned} H_I &= \frac{1}{2}[(\xi - \frac{g}{2\omega}(a_B - a_B^\dagger)^2)^2 + i\sqrt{2\omega}(a_B - a_B^\dagger)(\xi - \frac{g}{2\omega}(a_B - a_B^\dagger)^2)] \\ &+ \frac{ig}{\sqrt{2\omega}}(a_B - a_B^\dagger)[a_F, a_F^\dagger] \end{aligned} \tag{9.49}$$

The expectation value of this interaction Hamiltonian can now be calculated in the thermal vacuum using the relations in eqs. (9.41) and (9.42) and takes the form

$$\begin{aligned} \langle 0, \beta | H_I | 0, \beta \rangle &= \frac{1}{2}\left[\xi^2 + \frac{\xi g}{\omega}\frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}} + \frac{3g^2}{4\omega^2}\left(\frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}}\right)^2\right] \\ &= \frac{1}{2}\left[\left(\xi + \frac{g}{2\omega}\frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}}\right)^2 + \frac{g^2}{2\omega^2}\left(\frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}}\right)^2\right] \end{aligned} \tag{9.50}$$

This shows, as we would expect, that the thermal corrections to the ground state energy are explicitly positive, independent of the sign of the parameter ξ. The thermal vacuum, to first order in perturbation, has the energy given by

$$\begin{aligned} E(\beta) &= E_0(\beta) + \langle 0, \beta | H_I | 0, \beta \rangle \\ &= \omega\left(\frac{e^{-\beta\omega}}{1 - e^{-\beta\omega}} + \frac{e^{-\beta\omega}}{1 + e^{-\beta\omega}}\right) \end{aligned}$$

$$+\frac{1}{2}\left[\left(\xi+\frac{g}{2\omega}\frac{1+e^{-\beta\omega}}{1-e^{-\beta\omega}}\right)^2+\frac{g^2}{2\omega^2}\left(\frac{1+e^{-\beta\omega}}{1-e^{-\beta\omega}}\right)^2\right] \quad (9.51)$$

It is now clear that since the thermal effects are additive, interactions do not change our earlier conclusion that supersymmetry is spontaneously broken at finite temperature. We note, in particular from eq. (9.51), that at very hight temperatures, $T \to \infty$ ($\beta \to 0$),

$$\lim_{T\to\infty} E(\beta) \to \frac{3g^2}{2\beta^2\omega^4} + O(\beta^{-1}) \quad (9.52)$$

This shows that the dominant term in the ground state energy at very high temperatures is independent of the parameter ξ leading to the conclusion that independent of whether supersymmetry is broken at zero temperature or not, it must be broken at high temperatures. Although we have discussed the phenomenon of supersymmetry breaking in the context of a quantum mechanical model, it has also been studied in the context of supersymmetric quantum field theories and the results are unchanged.

9.4 Nicolai Map

The structure of supersymmetric theories is so restrictive that it leads to a very interesting map in such theories known as the Nicolai map. Very briefly, Nicolai map tells us that in a supersymmetric theory, it is possible to integrate out the fermions completely and absorb the resulting determinant into a redefinition of the bosonic fields such that the residual bosonic action can be written as a simple Gaussian.

This field redefinition cannot always be written in a closed form except in quantum mechanical models. In higher dimensional field theories, on the other hand, it can be constructed iteratively. Among other things, Nicolai map shows why the ground state energy in a supersymmetric theory has to vanish for all values of the coupling in the path integral formalism. Such a map exists and is meaningful only if the theory is supersymmetric. As a result, the existence of a Nicolai map becomes a measure of supersymmetry (order parameter). In this section, we will see that even the Nicolai map breaks down at finite temperature consistent with our earlier observation that supersymmetry is broken by thermal effects.

Before discussing the Nicolai map, let us note some identities involving the scalar and fermion propagators in quantum field theories. We know that, for a fermionic and a scalar field of the same mass m,

$$\begin{aligned} S(p,m) &= \frac{1}{\not{p} - m} \\ D(p,m) &= \frac{1}{p^2 - m^2} \end{aligned} \tag{9.53}$$

From the structure of these Greens functions, it follows now that

$$D(p,m) = -\frac{1}{n} \operatorname{Tr} S(p,m) S(-p,m) \tag{9.54}$$

where n is the number of space-time dimensions and "Tr" stands for the trace over the Dirac indices of the fermion fields. It is worth noting here that the identity in eq. (9.53) holds even when $m = 0$ and that for a $0+1$ dimensional field theory, namely, quantum mechanics, the

fermions do not have any Dirac index and consequently, the identity holds without the trace in the right hand side.

The identity in eq. (9.54) can also be written in several alternate ways in the coordinate space as

$$\begin{aligned} D(x-x') &= -\frac{1}{n}\,\mathrm{Tr}\int d^n x''\, S(x-x'')S(x'-x'') \\ D^{-1}(x-x') &= -\frac{1}{n}\,\mathrm{Tr}\int d^n x''\, S^{-1}(x-x'')S^{-1}(x'-x'') \\ \delta^n(x-x') &= -\frac{1}{n}\,\mathrm{Tr}\int d^n x_1 d^n x_2\, S(x_1-x)D^{-1}(x_1-x_2)S(x_2-x') \end{aligned} \tag{9.55}$$

As we know, a supersymmetric theory necessarily involves bosons and fermions of the same mass and the identities in eqs. (9.54) and (9.55) are crucial in establishing the Nicolai map in such theories.

Let us now, for simplicity, look at the supersymmetric quantum mechanical theory of eq. (9.14)

$$\begin{aligned} I &= \int dt\,[\frac{1}{2}((\dot{q})^2-(f(q))^2)+i\overline{\psi}\dot{\psi}-f'(q)\overline{\psi}\psi] \\ &= \int dtdt'\,[\frac{1}{2}(q(t')D^{-1}(t'-t)q(t)-\delta(t-t')(f(q))^2) \\ &\quad +\overline{\psi}(t')S^{-1}(t'-t)\psi(t)-\delta(t-t')f'(q)\overline{\psi}\psi] \end{aligned} \tag{9.56}$$

Here, we have identified

$$\begin{aligned} S^{-1}(t-t') &= i\frac{d}{dt}\,\delta(t-t') \\ D^{-1}(t-t') &= -\frac{d^2}{dt^2}\,\delta(t-t') \end{aligned} \tag{9.57}$$

corresponding to massless propagators assuming that if mass terms are present, they are all included in the interactions.

The integration over the fermionic variables can now be done in the path integral (the Lagrangian is bilinear in the fermionic fields) and would lead to

$$\det(S^{-1} - f'(q)) = \det S^{-1} \det(1 - Sf'(q)) \tag{9.58}$$

Furthermore, if we now redefine the bosonic variable as

$$q(t) \to q'(t) = q(t) - \int dt' S(t-t') f(q(t')) \tag{9.59}$$

or more compactly as

$$q \to q' = q - Sf(q) \tag{9.60}$$

then, we note that

$$\begin{aligned} &\int dt dt' q'(t') D^{-1}(t'-t) q'(t) \\ &\quad = q' D^{-1} q' = q D^{-1} q + f(q) S D^{-1} S f(q) \\ &\quad = \int dt dt' \left(q(t') D^{-1}(t'-t) q(t) - \delta(t-t')(f(q))^2 \right) \end{aligned} \tag{9.61}$$

Here we have used the identity involving the bosonic and fermionic propagators in eqs. (9.54) and (9.55) as well as the fact that for polynomials, $f(q)$, which do not support instantons, the crossed terms in eq. (9.61) integrate to zero.

We are now ready to show that the Nicolai map exists in this theory. In fact, we note that under the field redefinition in eq.

(9.60), the bosonic part of the action in eq. (9.56) does, indeed, become quadratic in the new variables. Moreover, the Jacobian resulting from the change of variables is given by

$$J = \left|\frac{\partial q'}{\partial q}\right| = \det(1 - Sf'(q)) \tag{9.62}$$

which is precisely the Matthews-Salam determinant resulting from the integration of the fermions in eq. (9.58). In other words, the fermion determinant can be absorbed into the bosonic field redefinition which makes the path integral a bosonic Gaussian integral, namely,

$$\begin{aligned} Z = e^{iW} &= \det S^{-1} \int \mathcal{D}q\, e^{\frac{i}{2}\int dt\, q' D^{-1} q'} \\ &= \det S^{-1} \left(\det(-D^{-1})\right)^{-1/2} \\ &= \left[\det(-SD^{-1}S)\right]^{-1/2} \\ &= 1 \end{aligned} \tag{9.63}$$

This relation is quite significant in the sense that it implies $W = 0$ which leads to the fact that the ground state energy in supersymmetric theories must vanish for all values of the coupling. As we have seen, in the path integral formalism, this follows from the existence of the Nicolai map which crucially depends on the identities involving the bosonic and the fermionic propagators.

Thus, we see that the Nicolai map, of course, depends on the very special structure of bosonic and fermionic interactions in supersymmetric theories, but it also crucially depends on the identities involving bosonic and fermionic propagators of the same mass. In fact,

it is clear that even in the absence of interactions, eq. (9.63) continues to hold because of the identities in eqs. (9.54) and (9.55). We will now show that these identities no longer hold at finite temperature leading to a breakdown of the Nicolai map. We can see this, in general, without restricting ourselves to the quantum mechanical case. We have already seen that in the real time formalism, the propagators separate into a temperature independent part and a temperature dependent part. Thus, for example, we see from eqs. (2.36) and (2.45) that the bosonic and the fermionic Greens functions can be written in the matrix form as

$$
\begin{aligned}
S(p,m) &= (\not{p}+m) \\
&\quad \times \left[\begin{pmatrix} \frac{1}{p^2-m^2+i\epsilon} & -2i\pi\theta(-p^0)\delta(p^2-m^2) \\ -2i\pi\theta(p^0)\delta(p^2-m^2) & -\frac{1}{p^2-m^2-i\epsilon} \end{pmatrix} \right. \\
&\quad \left. + 2i\pi n_F(|p^0|)\delta(p^2-m^2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] \\
D(p,m) &= \begin{pmatrix} \frac{1}{p^2-m^2+i\epsilon} & -2i\pi\theta(-p^0)\delta(p^2-m^2) \\ -2i\pi\theta(p^0)\delta(p^2-m^2) & -\frac{1}{p^2-m^2-i\epsilon} \end{pmatrix} \\
&\quad -2i\pi n_B(|p^0|)\delta(p^2-m^2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad (9.64)
\end{aligned}
$$

Here n_F and n_B represent the fermionic and the bosonic distribution functions respectively, namely,

$$
n_B(|p^0|) = \frac{1}{e^{\beta|p^0|}-1} \qquad n_F(|p^0|) = \frac{1}{e^{\beta|p^0|}+1}
$$

It is clear now that the temperature dependent parts of the propagators carry the statistics of the respective fields and since the distribution functions are distinctly different for the two classes of the fields, the zero temperature identities of eqs. (9.54) and (9.55) do not carry over to finite temperature.

At finite temperature, the structure of the Lagrangian does not change. Consequently, the special structure of the interactions in a supersymmetric theory continues to hold. However, the only temperature dependence comes in through the propagators which do not obey the identities necessary to show the existence of the Nicolai map. The Nicolai map, therefore, does not exist at finite temperature in a supersymmetric theory, consistent with our earlier conclusion that supersymmetry is spontaneously broken at finite temperature. It is worth noting here that even in the absence of interactions, eq. (9.63) does not hold at finite temperature because the propagators no longer satisfy the relations in eqs. (9.54) and (9.55). This shows that the ground state energy at finite temperature must be nonzero in a supersymmetric theory and, in some sense, clarifies the mechanism behind it.

9.5 Goldstino States

As we have seen in the earlier sections, supersymmetry is spontaneously broken at finite temperature. Furthermore, as we have noted in chapter 6 (see the discussion following eq. (6.20)), the spontaneous breakdown of a continuous global symmetry requires the existence of massless particles (Goldstone particles) in a manifestly Lorentz in-

variant theory with a positive Hilbert space. In the absence of manifest Lorentz invariance as is the case at finite temperature, on the other hand, we have seen in section 3.8, that Goldstone theorem only predicts zero energy singularities in the Greens functions. Thus, we extend "Goldstone states" (or "Goldstino states" in the case of the breakdown of a fermionic symmetry) to include such states.

In the case of the supersymmetric oscillator at finite temperature, we have explicitly seen that the "Goldstino states" consist of a "tilde" and a "nontilde" quanta as was already mentioned in chapter 3. More specifically, we note from eq. (9.45), that the Goldstino states are denoted by $|\chi_1\rangle$ and $|\chi_2\rangle$ and have the form

$$\begin{aligned} Q|0,\beta\rangle &\sim a_B^\dagger(\beta)\tilde{a}_F^\dagger(\beta)|0,\beta\rangle \sim |\chi_1\rangle \\ \overline{Q}|0,\beta\rangle &\sim a_F^\dagger(\beta)\tilde{a}_B^\dagger(\beta)|0,\beta\rangle \sim |\chi_2\rangle \end{aligned} \tag{9.65}$$

Therefore, we see that the "Goldstino states" consist of a thermal particle and a hole. Such a state is conventionally known as a super thermal pair. When supersymmetry is unbroken at zero temperature, it is a general feature that the "Goldstino states" at finite temperature consist of super thermal pairs.

This, however, raises an interesting question. Namely, as we have noted earlier, when supersymmetry is spontaneously broken at zero temperature, it is not restored at finite temperature, rather it continues to remain spontaneously broken. On the other hand, if supersymmetry is spontaneously broken at zero temperature, then, as

we know from the Goldstone theorem, there exists a massless fermionic state and, consequently, it is not clear whether this zero temperature state continues as the "Goldstino" state or a super thermal pair replaces it at finite temperature. In fact, at finite temperature, as we will see now, the zero temperature "Goldstino" state mixes with super thermal pair states and the true "Goldstino" state at finite temperature is a linear combination of these states while the orthogonal state becomes a massive state. Let us see this qualitatively. In general, we note that the supersymmetry generator being fermionic will have the zero temperature form

$$Q = \phi_B^\dagger \phi_F + v\psi_F^\dagger \tag{9.66}$$

where $\phi_B^\dagger$ and ϕ_F represent some bosonic creation and fermionic annihilation operators respectively. The supersymmetry breaking parameter is denoted by the vacuum expectation value v (which is in reality the vacuum expectation value of an auxiliary field of the theory) and $\psi_F^\dagger$ denotes the creation operator for the "Goldstino".

In terms of the thermal operators as is clear from the discussions of the earlier sections, however, the supersymmetry charge will have the form

$$Q = \alpha(\beta)\phi_B^\dagger(\beta)\tilde{\phi}_F^\dagger(\beta) + \gamma(\beta)v\psi_F^\dagger(\beta) + \cdots \tag{9.67}$$

where $\alpha(\beta)$ and $\gamma(\beta)$ are temperature dependent constants and "..." represent terms which contain at least one annihilation operator. It is clear now that the "Goldstino" state at finite temperature is given by

$$|\chi(\beta)\rangle = Q|0,\beta\rangle = \alpha(\beta)\phi_B^\dagger(\beta)\tilde{\phi}_F^\dagger(\beta)|0,\beta\rangle + \gamma(\beta)v\psi_F^\dagger(\beta)|0,\beta\rangle \tag{9.68}$$

We see, therefore, that if $v = 0$, namely, if supersymmetry is unbroken at zero temperature, the "Goldstino" state at finite temperature is given by a super thermal pair as we have already seen. On the other hand, if $v \neq 0$, *i.e.*, if supersymmetry is spontaneously broken at zero temperature, then the finite temperature "Goldstino" state is given by a linear combination of the zero temperature "Goldstino" state and a super thermal pair state. The orthogonal combination can be shown, with some work, to correspond to a massive state.

9.6 NonSimply Connected SpaceTime

We have seen in chapter 1 that, in the imaginary time formalism, a thermal field theory corresponds to defining the theory on a manifold $\mathbf{R}^3 \times S^1$ (see the discussion following eq. (1.40)). In many theories of current interest, it is common to define theories on non-simply connected space-time of this kind. For example, the higher dimensional Kaluza-Klein theories or string theories need to have several dimensions compactified (generally on a torus) and the prototype of such a compactification is on manifolds of the form $\mathbf{R}^n \times S^1$. Such theories are remarkably similar in structure to thermal field theories as we will see shortly. We have also seen earlier, in chapter 1 (see section 1.7) as well as in chapter 8, that at finite temperature the two point amplitude (as well as the higher ones) becomes nonanalytic at $p^\mu = 0$. However, we also know that in supersymmetric theories, there are remarkable cancellations among bosonic and fermionic loops and so, it is worth asking if such cancellations in a supersymmetric theory make the amplitudes analytic at finite temperature. There is, however,

an obvious difficulty in this investigation because, as we have seen in this chapter, a supersymmetric theory does not remain supersymmetric at finite temperature.

Theories defined on a non-simply connected space-time provide an alternative for such an investigation. Namely, although the structure of theories defined on a space-time $\mathbf{R}^3 \times S^1$ is similar to a thermal field theory, the compactification length L of S^1 is unrelated to temperature and the generating functional does not correspond to a partition function. Consequently, we are not required to impose periodic and anti-periodic boundary conditions for bosons and fermions respectively as we would in order to define a thermal partition function. The breaking of supersymmetry at finite temperature, as we have noted earlier, arises because of the different statistics (equivalently, periodicity conditions) satisfied by the bosons and fermions. In contrast, when we have a theory defined on a non-simply connected space-time, we are free to choose the periodicity conditions for the various fields along the direction being compactified. Consequently, we can choose the same periodicity conditions for bosons and fermions which is likely to preserve supersymmetry. In such a case, it is worth asking whether the amplitudes of the theory become analytic for a supersymmetric theory defined on, say $\mathbf{R}^3 \times S^1$, when the bosons and the fermions satisfy identical periodicity conditions along the compactified direction. In this section, we will first show that supersymmetry is, indeed, preserved when identical periodicity conditions are used for bosons and fermions in the sense that the zero point energy and the tadpoles vanish. Furthermore, we show that the nonanalyticity in the amplitudes continues to hold in spite of

supersymmetry – supersymmetry merely relates the non-analyticity in the bosonic and the fermionic amplitudes.

Let us consider the Wess-Zumino theory, which describes the supersymmetric interacting theory of Majorana fermions with a scalar and a pseudoscalar, described by the Lagrangian density

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu A\partial^\mu A + \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{m^2}{2}(A^2+B^2) + \frac{i}{2}\overline{\psi}\gamma^\mu\partial_\mu\psi - \frac{m}{2}\overline{\psi}\psi \\ &\quad -g\overline{\psi}(A - i\gamma_5 B)\psi - mgA(A^2+B^2) - \frac{g^2}{2}(A^2+B^2)^2 \qquad (9.69)\end{aligned}$$

Here A and B denote, respectively, the scalar and the pseudoscalar fields and ψ represents a Majorana (self charge conjugate) fermion. In $3+1$ dimensions, this theory has the special property (following from supersymmetry) that the tadpoles and the vacuum energy vanish. Furthermore, the radiative corrections to the two point functions have the form

$$= a(k^2+m^2)$$

$$= a(k^2+m^2)$$

$$= a\not{k} \qquad (9.70)$$

where a is a constant and the forms of the two point functions signify that the theory only needs a wave function renormalization. This is what we had alluded to earlier in that supersymmetric theories have much better renormalization properties than ordinary theories. These follow from the nonrenormalization theorem which we will not get into.

Suppose, we would like to study the Wess-Zumino theory on $\mathbf{R}^3 \times S^1$ where we assume that the coordinate x^3 is compactified with a compactification length L. We note here that this is different from the finite temperature case in that here a spatial coordinate is being compactified as opposed to the time coordinate in the Imaginary time formalism. The momentum along this axis (see eq. (1.29)) will be discrete and depending on the boundary condition will have the values

$$\left.\begin{aligned} p^3_{\text{per.}} &= \tfrac{2\pi n}{L} \\ p^3_{\text{anti.}} &= \tfrac{2\pi(n+1/2)}{L} \end{aligned}\right\} \quad n = 0, \pm 1, \pm 2, \cdots \tag{9.71}$$

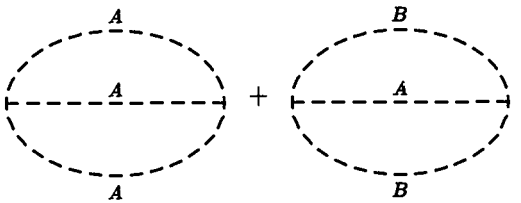
As discussed in chapter 1 (see eq. (1.43)), the four dimensional momentum integration will now be replaced by

$$\int \frac{d^4 p_E}{(2\pi)^4} \to \int \frac{d^3 p}{(2\pi)^3} \frac{1}{L} \sum_n \tag{9.72}$$

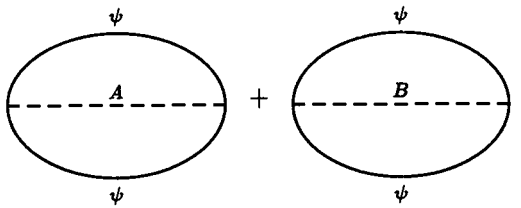
However, for simplicity, we will continue to use the notation of four dimensional integration with the understanding that it really stands for a three dimensional integral and a sum as described above.

With these notations, we are now ready to calculate the vacuum energy which, at the lowest order, arises at two loops in this

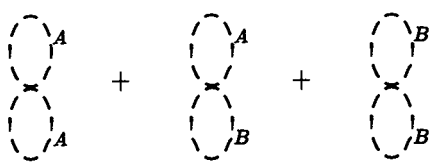
theory



$$= 4im^2g^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{1}{(k^2-m^2)(p^2-m^2)((k-p)^2-m^2)}$$



$$= -8ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{k \cdot p}{(k^2-m^2)(p^2-m^2)((k-p)^2-m^2)}$$



$$= 4ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{1}{(k^2-m^2)(p^2-m^2)} \tag{9.73}$$

It is now an easy matter to check that these three terms algebraically add up to zero (if both bosons and fermions satisfy identical boundary conditions) because of the symmetry properties of the integrals. This shows that the theory does not develop a vacuum energy even on $\mathbf{R}^3 \times S^1$ provided identical boundary conditions are used for the bosons and the fermions along the direction of compactification. Even though we have discussed the vanishing of the zero point energy only at two loops, this

analysis can be carried out order by order to show that the theory has a vanishing vacuum energy as is required by supersymmetry.

In a similar manner, we can also show that the theory does not develop any tadpoles when described on $\mathbf{R}^3 \times S^1$ for identical periodicity conditions for bosons and fermions. As a simple example, let us note that

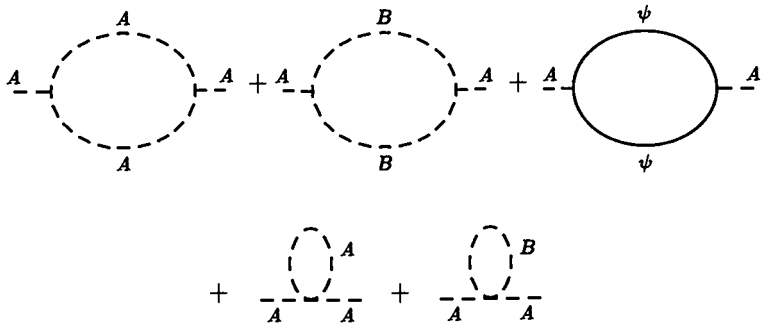
$$= 3mg \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2}$$

$$= mg \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2}$$

$$= -4mg \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} \tag{9.74}$$

Once again, for identical periodicity conditions, it is trivial to check that the three terms add up to zero leading to the conclusion that the theory does not develop any tadpoles, as would be required by supersymmetry.

Next, let us look at the bosonic and the fermionic two point functions of the theory. When the A, B and the ψ fields satisfy identical boundary conditions, the two point function for the A field, at one loop, becomes



$$= -i\pi_A(p) = 4g^2 \int \frac{d^4k}{(2\pi)^4} \frac{2p\cdot(k+p)+m^2}{(k^2-m^2)((k+p)^2-m^2)}$$

$$= 4g^2(p^2+m^2) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)((k+p)^2-m^2)} \qquad (9.75)$$

Here p^μ represents the external momentum. We write any four momentum, p^μ, as $(\hat{p}, p^3)$ where p^3 takes the values in eq. (9.71) depending on the periodicity conditions used for the fields.

We can now evaluate the two point function in eq. (9.75) explicitly using eqs. (9.71) and (9.72). Rotating to Euclidean space and using periodic boundary conditions, we obtain

$$-i\pi_A^{\text{per.}}(p) = -4ig^2(p_E^2-m^2) \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(k_E^2+m^2)((k_E+p_E)^2+m^2)}$$

$$= -4ig^2(p_E^2-m^2) \int \frac{d^3\hat{k}_E}{(2\pi)^3} \frac{1}{L}$$

$$\times \sum_n \frac{1}{(\hat{k}_E^2+(\frac{2\pi n}{L})^2+m^2)((\hat{k}_E+\hat{p}_E)^2+(p^3+\frac{2\pi n}{L})^2+m^2)} \qquad (9.76)$$

The sum in eq. (9.76) can be evaluated using eq. (1.46) and we can write (This is, in fact, the same integral as in eq. (1.51) with appropriate identification.)

$$-i\pi_A(p) = -i\pi_A^0(p) - i\pi_{A,L}^{\text{per.}}(p) \tag{9.77}$$

where π_A^0 is the part of the two point function independent of L (namely, the $L \to \infty$ value corresponding to the two point function in the four dimensional Wess-Zumino theory) and the L dependent part takes the form

$$\begin{aligned} -i\pi_{A,L}^{\text{per.}} &= -4ig^2(p_E^2 - m^2) \int \frac{d^3\hat{k}_E}{(2\pi)^3} \frac{n_B(L\omega_{\hat{k}})}{\omega_{\hat{k}}} \\ &\quad \times \left[\frac{1}{p_E^2 + 2(\hat{k}_E \cdot \hat{p}_E + ip^3\omega_{\hat{k}})} + \frac{1}{p_E^2 + 2(\hat{k}_E \cdot \hat{p}_E - ip^3\omega_{\hat{k}})} \right] \\ &= -\frac{ig^2}{2\pi^2} \frac{(p_E^2 - m^2)}{\hat{p}} \int_0^\infty d\hat{k}\, \frac{\hat{k}\, n_B(L\omega_{\hat{k}})}{\omega_{\hat{k}}} \ln R \end{aligned} \tag{9.78}$$

Here, we have defined,

$$\begin{aligned} \omega_{\hat{k}} &= (\hat{k}^2 + m^2)^{1/2} \\ R &= \frac{[p_E^2 + 2(\hat{k} \cdot \hat{p} + ip^3\omega_{\hat{k}})][p_E^2 + 2(\hat{k} \cdot \hat{p} - ip^3\omega_{\hat{k}})]}{[p_E^2 - 2(\hat{k} \cdot \hat{p} + ip^3\omega_{\hat{k}})][p_E^2 - 2(\hat{k} \cdot \hat{p} - ip^3\omega_{\hat{k}})]} \end{aligned} \tag{9.79}$$

We could have similarly evaluated the two point function for the A field with anti-periodic boundary condition for all the fields. Here the appropriate identity to use is

$$\sum_n f(2n+1) = \sum \pi \,\text{Res} f(z) \tan \pi z \quad \text{at the poles of } f(z)$$

and without going into details, we simply note here the result that in such a case, we can write

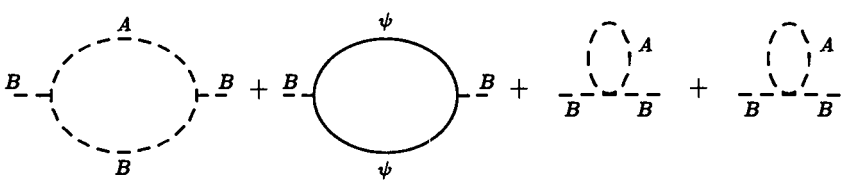
$$-i\pi_A^{\text{anti.}}(p_E) = -i\pi_A^0 - i\pi_{A,L}^{\text{anti.}} \tag{9.80}$$

where

$$-i\pi_{A,L}^{\text{anti.}}(p_E) = \frac{ig^2}{2\pi^2}\frac{(p_E^2 - m^2)}{\hat{p}}\int_0^\infty d\hat{k}\frac{\hat{k}\, n_F(L\omega_{\hat{k}})}{\omega_{\hat{k}}}\ln R \tag{9.81}$$

with n_F representing the fermionic distribution function.

The two point functions for the B field can similarly be obtained and we simply note here the results

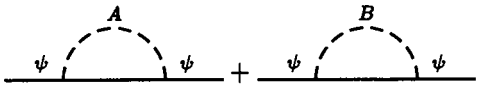


$$= -i\pi_B(p_E) = -i\pi_B^0(p_E) - i\pi_{B,L}(p_E) \tag{9.82}$$

with

$$\begin{aligned} -i\pi_{B,L}^{\text{per.}}(p_E) &= -\frac{ig^2}{2\pi^2}\frac{(p_E^2 - m^2)}{\hat{p}}\int_0^\infty d\hat{k}\,\frac{\hat{k}\, n_B(L\omega_{\hat{k}})}{\omega_{\hat{k}}}\ln R \\ -i\pi_{B,L}^{\text{anti.}}(p_E) &= \frac{ig^2}{2\pi^2}\frac{(p_E^2 - m^2)}{\hat{p}}\int_0^\infty d\hat{k}\frac{\hat{k}\, n_F(L\omega_{\hat{k}})}{\omega_{\hat{k}}}\ln R \end{aligned} \tag{9.83}$$

The two point function for the fermion field can also be calculated in a straightforward manner and the result is



$$= -i\pi_\psi(p_E) = -i\pi^0_\psi(p_E) - i\pi_{\psi,L}(p_E) \tag{9.84}$$

where

$$\begin{aligned} -i\pi^{\text{per.}}_{\psi,L}(p_E) &= -\frac{ig^2}{2\pi^2}\frac{\not{p}_E}{\hat{p}}\int_0^\infty d\hat{k}\frac{\hat{k}n_B\left(L\omega_{\hat{k}}\right)}{\omega_{\hat{k}}}\ln R \\ -i\pi^{\text{anti.}}_{\psi,L}(p_E) &= \frac{ig^2}{2\pi^2}\frac{\not{p}_E}{\hat{p}}\int_0^\infty d\hat{k}\frac{\hat{k}n_F\left(L\omega_{\hat{k}}\right)}{\omega_{\hat{k}}}\ln R \end{aligned} \tag{9.85}$$

Thus, we see that, at one loop, the three two point functions can be written as

$$\begin{aligned} -i\pi_{A,L}(p_E) &= (p_E^2 - m^2)\pi_L(p_E) \\ -i\pi_{B,L}(p_E) &= (p_E^2 - m^2)\pi_L(p_E) \\ -i\pi_{\psi,L}(p_E) &= \not{p}_E\pi_L(p_E) \end{aligned} \tag{9.86}$$

where we have

$$\begin{aligned} \pi^{\text{per.}}_L(p_E) &= -\frac{ig^2}{2\pi^2\hat{p}}\int_0^\infty d\hat{k}\frac{\hat{k}\,n_B(L\omega_{\hat{k}})}{\omega_{\hat{k}}}\ln R \\ \pi^{\text{anti.}}_L(p_E) &= \frac{ig^2}{2\pi^2\hat{p}}\int_0^\infty d\hat{k}\frac{\hat{k}\,n_F(L\omega_{\hat{k}})}{\omega_{\hat{k}}}\ln R \end{aligned} \tag{9.87}$$

We note here that when rotated to Minkowski space, the forms of the two point functions are consistent with the nonrenormalization theorem as discussed in eq. (9.70). However, it is also clear

that the two point functions become nonanalytic at the origin in the momentum space. This can be seen from the fact that

$$\begin{aligned}\lim_{\hat{p}\to 0\, p^3\to 0} \pi_L^{\text{per.}}(p_E) &= -\frac{ig^2}{\pi^2}\int_0^\infty d\hat{k}\,\frac{n_B(L\omega_{\hat{k}})}{\omega_{\hat{k}}} \\ \lim_{\hat{p}\to 0\, p^3\to 0} \pi_L^{\text{anti.}}(p_E) &= \frac{ig^2}{\pi^2}\int_0^\infty d\hat{k}\,\frac{n_F(L\omega_{\hat{k}})}{\omega_{\hat{k}}}\end{aligned} \tag{9.88}$$

On the other hand, when the limits are reversed, the expression becomes

$$\begin{aligned}\lim_{p^3\to 0\,\hat{p}\to 0} \pi_L^{\text{per.}}(p_E) &= -\frac{ig^2}{\pi^2}\int_0^\infty d\hat{k}\,\frac{\hat{k}^2 n_B(L\omega_{\hat{k}})}{\omega_{\hat{k}}^3} \\ \lim_{p^3\to 0\,\hat{p}\to 0} \pi_L^{\text{anti.}}(p_E) &= \frac{ig^2}{\pi^2}\int_0^\infty d\hat{k}\,\frac{\hat{k}^2 n_F(L\omega_{\hat{k}})}{\omega_{\hat{k}}^3}\end{aligned} \tag{9.89}$$

This nonanalytic behavior is precisely the same as seen at finite temperature (see eqs. (1.57) and (1.58)). Supersymmetry has not improved this behavior. If anything, it has only related the nonanalyticity in the two point functions for the A, B and the ψ fields. We merely note here without going into details that a periodic boundary condition for the fermionic fields is known to lead to acausal behavior. In such a case, the only consistent boundary condition that can be used for both the bosonic and the fermionic fields appears to be the anti-periodic boundary condition for both bosons and fermions which is what should be used in a consistent compactification of higher dimensional theories.

9.7 References

Das, A, Physica **A158** (1989) 1.

Das, A and **M. Hott**, Mod. Phys. Lett. **A10** (1995) 893.

Das, A and **M. Kaku**, Phys. Rev. **D18** (1978) 4540.

Das, A, A. Kharev and **V.S. Mathur**, Phys. Lett. **181B** (1986) 299.

Das, A and **V.S. Mathur**, Phys. Rev. **D35** (1987) 2053.

Das, A and **V.S. Mathur**, Indian J. Phys. **B61** (1987) 214.

Ford, L, Phys. Rev. **D20** (1980) 933.

Matsumoto, H, M. Nakahara, Y. Nakano and **H. Umezawa**, Phys. Rev. **D29** (1984) 2838.

Nicolai, H, Nuc. Phys. **B170** (1980) 419.

Witten, E, Nuc. Phys. **B202** (1982) 253.

Chapter 10

Derivative Expansion

The low energy properties of a fundamental theory can be quite profitably studied through the derivative expansion. The basic idea behind derivative expansion is quite simple. When a theory contains a mass scale which is large compared to the energy scales at which we are interested, we can define an effective theory by integrating out these heavy fields. The resulting effective theory can, then, be expanded in a power series in the ratio of the energy scale and the heavy mass scale. Clearly, if one is interested in processes involving energy scales much smaller than the heavy mass scale, then, only a few terms in the expansion will be able to describe the low energy phenomena quite successfully. This method of expansion in powers of momentum or derivatives is commonly known as the derivative expansion. One can, equivalently, think of this also as an inverse (heavy) mass expansion.

The method of derivative expansion has been used, in this way, in the study of decays such as $\pi^0 \to 2\gamma$ as well as in the study of the properties of Skyrmions, just to name a few. It can also be used to study properties such as superconductivity, superfluidity etc. Clearly, it makes sense to make an expansion in powers of momentum only if there is a heavy mass in the theory. In the absence of such a mass scale, it is known that such an expansion is plagued with infrared divergences which leads to a breakdown of the derivative expansion. However, in some simple two dimensional models, even in the absence of a heavy mass, we will see that the method of derivative expansion works, namely, even though every term in the expansion is infrared divergent, the divergences cancel when the series is summed.

10.1 Derivative Expansion

The method of derivative expansion can be best described with the example of a fermion field with a heavy mass interacting with other fields which are not as massive. Thus, for example, we can consider the fermion Lagrangian density to be of the form

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m - K(\phi))\psi = \overline{\psi}(S_F^{-1} - K(\phi))\psi \tag{10.1}$$

where the fermion mass, m, is assumed to be heavy. $K(\phi)$ represents the interaction of some generic bosonic field, ϕ (which is not necessarily a scalar field), with the fermions while S_F is the fermionic Greens function defined to be (The Feynman "$i\epsilon$" prescription is understood)

$$S_F = \frac{1}{i\partial\!\!\!/ - m} \tag{10.2}$$

which, in the momentum space, takes the form

$$S_F(p) = \frac{1}{\not{p} - m} \tag{10.3}$$

The fermionic Lagrangian density in eq. (10.1) is bilinear in the fermion fields which can be integrated out in a straightforward manner in the path integral formalism to yield a determinant, namely,

$$\begin{aligned} Z(\phi) &= \int \mathcal{D}\psi \mathcal{D}\overline{\psi}\, e^{i\int d^4x\, \mathcal{L}} \\ &= N \det(S_F^{-1} - K(\phi)) \\ &= N(\det S_F^{-1})\,[\det(1 - S_F K(\phi))] \\ &= N' \det(1 - S_F K(\phi)) \end{aligned} \tag{10.4}$$

Here N is a normalization constant and in the last line we have absorbed an uninteresting constant into the normalization constant to write the generating functional in terms of the Matthews-Salam determinant. We see, therefore, that integrating out the fermion fields yields an effective action for the bosonic field ϕ of the form

$$\begin{aligned} Z(\phi) = e^{iS_{eff}(\phi)} &= N' e^{\mathrm{Tr}\ln(1 - S_F K(\phi))} \\ \text{or,} \quad S_{eff}(\phi) &= -i\,\mathrm{Tr}\ln(1 - S_F K(\phi)) \end{aligned} \tag{10.5}$$

where we have neglected an uninteresting constant in writing the effective action for the bosonic field and used the fact that for any matrix, A,

$$\det A = e^{\mathrm{Tr}\ln A}$$

where "Tr " represents a trace over any matrix index the fermion fields may have (Dirac indices, internal symmetry indices etc.) including an integration over space-time indices.

In general, the expression on the right hand side of eq. (10.5) cannot be evaluated in a closed form. However, when the fermion field is massive, we can expand it as a series in powers of the derivatives in the following way. We note first that for any matrix, A, we can expand the logarithm as usual as

$$\ln(1-A) = -A - \frac{A^2}{2} - \frac{A^3}{3} + \cdots$$

Moreover, if we think of the Greens function, $S_F(p)$, as a momentum dependent operator and $K(\phi(x))$ as a coordinate dependent operator, then, the "Tr " can be simply written as

$$\mathrm{Tr}A = \mathrm{tr}\int d^4x \, \langle x|A|x\rangle \tag{10.6}$$

where "tr " stands for the trace over other indices such as the Dirac index and the internal symmetry indices. With these, we can expand the right hand side of eq. (10.5) as

$$\begin{aligned}\mathrm{Tr}\ln(1-S_FK(\phi)) &= -\mathrm{tr}\int d^4x \, \langle x|[S_F(p)K(\phi(x)) \\ &+ \frac{S_F(p)K(\phi(x))S_F(p)K(\phi(x))}{2} + \cdots]|x\rangle\end{aligned} \tag{10.7}$$

The expression in eq. (10.7) is not yet in a form for direct evaluation because the momentum and the coordinate operators

do not commute. However, we can move the momentum dependent factors past the coordinate dependent ones through the use of various identities such as (Note from eqs. (10.2) and (10.3) the identification $p_\mu \to i\partial_\mu$.)

$$\begin{aligned}
[p_\mu, g(x)] &= i(\partial_\mu g(x)) \\
\left[p^2, g(x)\right] &= (\Box g(x)) + 2ip^\mu(\partial_\mu g(x)) \\
\left[\frac{1}{p^2}, g(x)\right] &= \frac{1}{(p^2)^2}\left[p^2, g(x)\right] + \frac{1}{(p^2)^3}\left[p^2, \left[p^2, g(x)\right]\right] + \cdots
\end{aligned} \tag{10.8}$$

While the first two identities in eq. (10.8) are straightforward, the third can be obtained from the series expansion of the integral representation

$$\frac{1}{p^2} = \int_0^\infty d\beta\, e^{-\beta p^2}$$

With the help of these identities (and others following from these), all the momentum dependent terms in eq. (10.7) can be moved past the coordinate dependent terms so that all the coordinate dependent terms are on the right (or left) and each term in the expression in eq. (10.7) can be written in the form

$$\begin{aligned}
\mathrm{tr} \int d^4x\, \langle x|A(p)B(x)|x\rangle &= \mathrm{tr} \int d^4x d^4p\, \langle x|A(p)|p\rangle\langle p|B(x)|x\rangle \\
&= \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} A(p) \int d^4x\, B(x)
\end{aligned} \tag{10.9}$$

Here we have used the fact that the inner product of the coordinate and the momentum basis states merely gives the phase factor

$$\langle x|p\rangle = \frac{e^{-ip\cdot x}}{(2\pi)^2}$$

Furthermore, the integrands in eq. (10.9) are ordinary functions and the momentum integration would lead to a multiplicative factor while the term with the coordinate integration has exactly the form of an action. Comparing with eqs. (10.5) and (10.7), we see now that integrating out the fermion fields leads to an effective action for the bosonic fields which would be a series in powers of the derivatives as is clear from the identities in eq. (10.8). This is the spirit of the derivative expansion and even though we have discussed this within the context of interacting fermions, it works equally well in other theories.

10.2 Soluble Models

We have discussed the general idea behind the derivative expansion in the last section. In this section, we will describe how it works in the context of simple two dimensional models which are known to be soluble. This would not only clarify how the method works, but would also bring out the interesting feature that in some models without a heavy mass, the derivative expansion still works without any problem of infrared divergence. Let us consider, in $1+1$ dimensions, fermions interacting with an Abelian gauge field. The Lagrangian density for the fermions is given by (see eqs. (4.10) and (4.11))

$$\mathcal{L}_f = i\overline{\psi}\gamma^\mu D_\mu\psi = \overline{\psi}\gamma^\mu(i\partial_\mu + gA_\mu)\psi \tag{10.10}$$

and the action in two dimensions takes the form

$$S_f = \int d^2x\, \mathcal{L}_f \tag{10.11}$$

The fermions are massless and it is easy to check that the gauge coupling, g, in eq. (10.10) has the canonical dimension of a mass.

In two dimensions, the Dirac gamma matrices satisfy various identities. Namely, with $\eta^{\mu\nu} = (+,-)$, γ^0 Hermitian and γ^1 anti-Hermitian, if we define a Hermitian $\gamma_5 = \gamma^0\gamma^1$, then it is easy to check that ($(\gamma^0)^2 = 1$, $(\gamma^1)^2 = -1$ and $(\gamma_5)^2 = 1$)

$$\begin{aligned} \gamma^\mu\gamma^\nu &= \eta^{\mu\nu} + \epsilon^{\mu\nu}\gamma_5 \\ \epsilon^{\mu\nu}\gamma_\nu &= \gamma_5\gamma^\mu \end{aligned} \tag{10.12}$$

where the totally anti-symmetric tensor, $\epsilon^{\mu\nu}$, is the two dimensional Levi-Civita tensor, satisfying $\epsilon^{01} = 1$. Furthermore, the dual of a vector in two dimensions is also a vector, defined as

$$A^5_\mu = \epsilon_{\mu\nu}A^\nu \tag{10.13}$$

and, therefore, we can write the Lagrangian density of eq. (10.11) also as

$$\begin{aligned} \mathcal{L}_f &= i\overline{\psi}\gamma^\mu D_\mu\psi \\ &= \overline{\psi}\gamma^\mu(i\partial_\mu + g\xi A_\mu + g\eta\gamma_5 A^5_\mu)\psi \qquad \xi + \eta = 1 \end{aligned} \tag{10.14}$$

As is well known, different values of the parameters ξ and η correspond to different regularizations used to study the theory in the path integral formalism. In particular, $\xi = 1$ and $\eta = 0$ corresponds to a gauge invariant regularization while $\xi = 0$ and $\eta = 1$ corresponds to a regularization which preserves the axial invariance.

The generating functional, as we have seen in eq. (10.5), is given by

$$Z[A_\mu] = e^{iS_{eff}[A_\mu]} = \int \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\, e^{iS_f} \tag{10.15}$$

and is best evaluated by rotating to Euclidean space where the path integrals are well behaved. In rotating to Euclidean space, we treat both ψ and $\overline{\psi}$ as independent fields, $x^0 \to -ix_2$, $\partial_0 \to i\partial_2$ and $\gamma^0 \to i\gamma_2$. Consequently, $\gamma_5 = \gamma^0\gamma^1 \to i\gamma_1\gamma_2$ continues to be Hermitian in Euclidean space although individual γ_μ's are anti-Hermitian. The identity of eq. (10.12), in Euclidean space, takes the form

$$\gamma_\mu\gamma_\nu = -\delta_{\mu\nu} - i\epsilon_{\mu\nu}\gamma_5 \tag{10.16}$$

where the Levi-Civita tensor in the Euclidean space has the value $\epsilon_{12} = 1$. We also note that, for the covariant derivative to transform as the ordinary derivative, namely, for $D_0 \to iD_2$, we must have $A_0 \to iA_2$ as well as $A_0^5 \to iA_2^5$. This shows that it is impossible to maintain the reality of A_μ and A_μ^5 in the Euclidean space simultaneously with the duality between them (see eq. (10.13)). Therefore, we choose to treat them as independent real fields in the Euclidean space. After all the calculations are done, and we have rotated back to Minkowski space, we can use the duality between these two fields, namely, eq. (10.13).

With all these relations, we note that the action for the fermion fields, in Euclidean space, becomes

$$S_f = i\int d^2x\, \overline{\psi}\gamma_\mu(i\partial_\mu + g\xi A_\mu + g\eta\gamma_5 A_\mu^5)\psi \tag{10.17}$$

and following eq. (10.5) we can write down the generating functional to have the form

$$Z[A_\mu] = N' \det(1 + S_F(g\xi \not{A} - g\eta\gamma_5 \not{A}^5)) \tag{10.18}$$

so that the Euclidean effective action takes the form (see eq. (10.5))

$$\begin{aligned} S_{eff}[A_\mu] &= \mathrm{Tr} \ln (1 + S_F(g\xi \not{A} - g\eta\gamma_5 \not{A}^5)) \\ &= \mathrm{Tr} \sum_{n=0} \frac{(-1)^n}{n + \frac{1}{2}} \left(S_F(g\xi \not{A} - g\eta\gamma_5 \not{A}^5)\right)^{n+1} \end{aligned} \tag{10.19}$$

In the present case, the fermion fields are massless and, consequently, S_F has the simple form

$$S_F(p) = \frac{1}{\not{p}} = \frac{\not{p}}{p^2} \tag{10.20}$$

It is easy to show that in $1 + 1$ dimensions, the series in eq. (10.19) involves only a finite number of terms. To see this, let us define

$$\not{B} = g(\xi \not{A} - \eta\gamma_5 \not{A}^5) \tag{10.21}$$

and note that in $1 + 1$ dimension, any vector has the unique decomposition into a longitudinal part and a transverse part as

$$B_\mu = \partial_\mu \phi(x) + \epsilon_{\mu\nu} \partial^\nu \rho(x) \tag{10.22}$$

Therefore, we can write (We will show the vanishing of the higher order terms in Minkowski space, for simplicity.)

$$\begin{aligned} S_F(p)(g\xi \not{A} - g\eta\gamma_5 \not{A}^5) &= \frac{1}{\not{p}} \not{B} \\ &= \frac{1}{\not{p}} (\not{\partial} \phi(x) - \gamma_5 \not{\partial} \rho(x)) \end{aligned} \tag{10.23}$$

where we have used eq. (10.12). Furthermore, using the commutation relations in eq. (10.8), we can write

$$\begin{aligned}\frac{1}{\not p}\not B &= \frac{1}{\not p}(\not\partial\phi(x) - \gamma_5\not\partial\rho(x)) \\ &= \frac{1}{\not p}(-i)\,[\not p, \phi(x)] + \gamma_5\frac{1}{\not p}(-i)\,[\not p, \rho(x)] \\ &= -i\phi(x) + i\frac{1}{\not p}\phi(x)\not p - i\gamma_5\rho(x) + i\gamma_5\frac{1}{\not p}\rho(x)\not p \qquad (10.24)\end{aligned}$$

As a result, we can write

$$\begin{aligned}\frac{1}{\not p}\not B\frac{1}{\not p}\not B &= \left(-i\phi(x)\frac{1}{\not p} + i\frac{1}{\not p}\phi(x) - i\gamma_5\rho(x)\frac{1}{\not p} + i\gamma_5\frac{1}{\not p}\rho(x)\right)\not B \\ &= -i\left[\phi(x), \frac{1}{\not p}\not B\right] - i\gamma_5\left[\rho(x), \frac{1}{\not p}\not B\right] \\ &= -i\left[\phi(x) + \gamma_5\rho(x), \frac{1}{\not p}\not B\right] \qquad (10.25)\end{aligned}$$

It is clear now, from eq. (10.25), that, for $n \geq 1$,

$$\left(\frac{1}{\not p}\not B\right)^{n+1} = -\frac{i}{n}\left[\phi(x) + \gamma_5\rho(x), \left(\frac{1}{\not p}\not B\right)^n\right] \qquad (10.26)$$

which leads to the fact that

$$\mathrm{Tr}\left(\frac{1}{\not p}\not B\right)^{n+1} = -\frac{i}{n}\mathrm{Tr}\left[\phi(x) + \gamma_5\rho(x), \left(\frac{1}{\not p}\not B\right)^n\right] \qquad (10.27)$$

From simple power counting, we note that in $1+1$ dimension, the momentum integrals would be divergent only for $n = 0, 1$. For $n > 1$, on the other hand, the integrals are convergent and consequently we can use the cyclicity properties of "Tr " to conclude that

$$\mathrm{Tr}\left(\frac{1}{\not p}\not B\right)^{n+1} = 0 \qquad \text{for } n > 1 \tag{10.28}$$

This shows that the series in eq. (10.19) contains only two terms

$$\begin{aligned} S_{eff} &= \mathrm{Tr}\left(g\frac{\not p}{p^2}(\xi \not A - \eta\gamma_5 \not A^5)\right. \\ &\quad \left. -\frac{g^2}{2}\frac{\not p}{p^2}(\xi \not A - \eta\gamma_5 \not A^5)\frac{\not p}{p^2}(\xi \not A - \eta\gamma_5 \not A^5)\right) \end{aligned} \tag{10.29}$$

Furthermore, the first term in the series vanishes because it is odd in the momentum variable (and, therefore, the momentum integral would vanish) and, consequently, the series contains only a single term which is quadratic in the field variables. Using the commutation relations in eq. (10.8), we can simplify this to be

$$\begin{aligned} S_{eff} &= -\frac{g^2}{2}(\mathrm{tr}\ \gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho)\int d^2x \frac{d^2p}{(2\pi)^2}\frac{p_\mu(p_\lambda - i\partial_\lambda)}{p^2(p-i\partial)^2} \\ &\quad \times(\xi^2 A_\nu(x)A_\rho(x) + \eta^2 A^5_\nu(x)A^5_\rho(x)) \\ &\quad -\frac{g^2\xi\eta}{2}(\mathrm{tr}\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho)\int d^2x \frac{d^2p}{(2\pi)^2}\frac{p_\mu(p_\lambda - i\partial_\lambda)}{p^2(p-i\partial)^2} \\ &\quad \times(A_\nu(x)A^5_\rho(x) + A^5_\nu(x)A_\rho(x)) \end{aligned} \tag{10.30}$$

Here the derivatives are understood to act only on the first of the two fields. It is worth emphasizing here that normally, in derivative expansion, one expands in powers of the derivatives to obtain an infinite series at every order of expansion. Here, in contrast, we have summed the infinite series to obtain the simple expression above. Furthermore, we

have separated the Dirac trace which will be done after the momentum integrals are evaluated using dimensional regularization to give

$$\begin{aligned} S_{eff} &= -\frac{g^2}{2\pi}\int d^2x \left[\xi^2\left(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)A_\mu(x)A_\nu(x)\right. \\ &\quad +\eta^2\left(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)A^5_\mu(x)A^5_\nu(x) \\ &\quad -\frac{\xi\eta}{2}\left(\epsilon_{\lambda\mu}\frac{\partial_\nu\partial_\lambda}{\Box}+\epsilon_{\lambda\nu}\frac{\partial_\mu\partial_\lambda}{\Box}\right)A_\mu(x)A^5_\nu(x) \\ &\quad \left.-\frac{\xi\eta}{2}\left(\epsilon_{\lambda\mu}\frac{\partial_\nu\partial_\lambda}{\Box}+\epsilon_{\lambda\nu}\frac{\partial_\mu\partial_\lambda}{\Box}\right)A^5_\mu(x)A_\nu(x)\right] \end{aligned} \tag{10.31}$$

It is understood here that the derivatives act only on the first of the two field variables.

In Euclidean space, A_μ and A^5_μ are treated as independent. However, now that the integrals are evaluated, we can rotate everything back to Minkowski space where we can use the duality relation, eq. (10.13), between A_μ and A^5_μ to write the effective action as

$$\begin{aligned} S_{eff} &= \frac{g^2}{2\pi}\int d^2x\, A_\mu(x)\left(\xi\eta^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{\Box}\right)A_\nu(x) \\ &= -\frac{g^2}{2\pi}\int d^2x\, A_\mu(x)(\eta\eta^{\mu\alpha}\eta^{\nu\beta}+\xi\epsilon^{\mu\alpha}\epsilon^{\nu\beta})\frac{\partial_\alpha\partial_\beta}{\Box}A_\nu(x) \end{aligned} \tag{10.32}$$

It is clear now that different values of ξ and η (such that $\xi+\eta=1$) lead to different effective actions corresponding to distinct regularizations. In particular when $\xi=1$ and $\eta=0$, the effective action is obviously gauge invariant.

We can now study various soluble two dimensional models directly from eq. (10.32). Let us consider the massive vector boson model whose Lagrangian density is given by

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu^2}{2}A_\mu A^\mu + \mathcal{L}_f \\ &= \mathcal{L}[A_\mu] + \mathcal{L}_f \end{aligned} \tag{10.33}$$

where μ is the mass of the vector boson and $\mathcal{L}_f$ represents the fermion Lagrangian density of eq. (10.10). In this case, the Euler-Lagrange equation for the vector field gives

$$\partial_\nu F^{\mu\nu} = \mu^2 A^\mu \tag{10.34}$$

whose consistency requires that

$$\partial_\mu A^\mu = 0 \tag{10.35}$$

Once again, integrating out the fermion fields, subject to the condition of eq. (10.35), would yield an effective Lagrangian density for the gauge fields which can be obtained from eq. (10.32) so that the resulting Lagrangian density for the vector field becomes (remember $\partial_\mu A^\mu = 0$ from eq. (10.35).)

$$\mathcal{L}_{TOT} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m_R^2}{2}A_\mu A^\mu \tag{10.36}$$

We see that the fermion interactions merely lead to a renormalization of the vector boson mass as

$$m_R^2 = \mu^2 + \frac{\xi g^2}{2\pi} \tag{10.37}$$

In particular, we note that in the case of the Schwinger model where the gauge boson is massless, *i.e.*, $\mu = 0$, a gauge invariant regularization ($\xi = 1$) would lead to a gauge boson mass

$$m_R^2 = \frac{g^2}{2\pi} \tag{10.38}$$

which is the standard result.

In a similar manner, we can also evaluate the generating functional for the massless Thirring model as follows. The Thirring model represents a self-interacting fermion theory in $1 + 1$ dimension and is described by the Lagrangian density

$$\begin{aligned} \mathcal{L} &= i\overline{\psi}\partial\!\!\!/\psi - \frac{\lambda}{2}\overline{\psi}\gamma^\mu\psi\overline{\psi}\gamma_\mu\psi + \overline{\psi}\gamma^\mu\psi A_\mu \\ &= \mathcal{L}_f - \frac{\lambda}{2}\overline{\psi}\gamma^\mu\psi\overline{\psi}\gamma_\mu\psi \end{aligned} \tag{10.39}$$

Here we have added an external source, A_μ, to the Lagrangian density (to derive correlation functions) and $\mathcal{L}_f$ represents the Lagrangian density of eq. (10.10). The generating functional for the Thirring model can now be written as

$$\begin{aligned} Z_{TH} &= \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{i\int d^2x(\mathcal{L}_f - \frac{\lambda}{2}\overline{\psi}\gamma^\mu\psi\overline{\psi}\gamma_\mu\psi)} \\ &= \exp\left(\frac{i\lambda}{2}\int d^2x \,\frac{\delta^2}{\delta A_\mu(x)\delta A^\mu(x)}\right)\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{i\int d^2x\,\mathcal{L}_f} \end{aligned} \tag{10.40}$$

The generating functional for the fermion fields is already evaluated in eq. (10.32). Substituting this into eq. (10.40) yields the generating functional for the Thirring model to be

$$Z_{TH} = N \exp\left(-\frac{i}{2\pi}\int d^2x\, A_\mu \left[\frac{\eta\eta^{\mu\alpha}\eta^{\nu\beta}}{1-\lambda\eta/\pi} + \frac{\xi\epsilon^{\mu\alpha}\epsilon\nu\beta}{1+\lambda\xi/\pi}\right]\frac{\partial_\alpha\partial_\beta}{\Box}A_\nu\right) \tag{10.41}$$

Other simple two dimensional models can also be solved in this manner and the important thing to note here is that even without a heavy mass for fermions, the derivative expansion works without any problems of infrared divergence.

10.3 Derivative Expansion at $T \neq 0$

As we have seen earlier, there are novel features that develop at finite temperature. In particular, we have seen that at finite temperature, Feynman amplitudes become nonanalytic at the origin in the momentum space. It is, therefore, of interest to investigate whether the derivative expansion continues to hold at finite temperature. To examine this, we will study a toy model (in $3+1$ dimensions) of a heavy scalar field interacting with a lighter scalar field through a cubic term described by the Lagrangian density

$$\mathcal{L}(B,\phi) = \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{M^2}{2}B^2 - \frac{g}{2}\phi B^2 + \mathcal{L}_0(\phi) \tag{10.42}$$

We assume that $\mathcal{L}_0(\phi)$ represents the free Lagrangian density for the lighter scalar field, ϕ, and M denotes the heavy mass associated with the scalar field B. Clearly, this is an unrealistic model. However, this has been extensively studied in connection with the nonanalyticity properties at finite temperature (In fact, this is also the model we have analyzed in chapter 1, see eq. (1.50).) and, therefore, it is meaningful to study this model in connection with derivative expansion.

Since the theory is at most quadratic in the heavy B field, we can integrate it out in the path integral to obtain an effective action for the lighter ϕ field as

$$\begin{aligned} Z &= \int \mathcal{D}\phi\mathcal{D}B\, e^{iS(B,\phi)} \\ &= \int \mathcal{D}\phi\, \left(\det(G_F^{-1}(p) - g\phi(x))\right)^{-\frac{1}{2}}\, e^{iS_0(\phi)} \\ &= N\int \mathcal{D}\phi\, (\det(1 - gG_F(p)\phi(x)))^{-\frac{1}{2}}\, e^{iS_0(\phi)} \\ &= N\int \mathcal{D}\phi\, e^{i(S_0(\phi)+S_{eff}(\phi))} \\ &= N\int \mathcal{D}\phi\, e^{iS_{TOT}(\phi)} \end{aligned} \tag{10.43}$$

where we have absorbed an uninteresting determinant factor into the normalization constant (the power of the scalar determinant is different from that for the fermions) and with periodic boundary conditions for the B field, the scalar Greens function at finite temperature has the form (see eq. (2.36))

$$\begin{aligned} G_F(p) &= \frac{1}{p^2 - M^2 + i\epsilon} - 2i\pi n_B(|p^0|)\delta(p^2 - M^2) \\ &= G_F^0(p) + G_F^\beta(p) \end{aligned} \tag{10.44}$$

with the bosonic distribution function given by

$$n_B(|p^0|) = \frac{1}{e^{\beta|p^0|} - 1}$$

We note here that for one loop effects, as is the case here, we need only the $++$ component of the propagator.

As before, we note that the integration of the heavy B field gives rise to an effective scalar action of the form (The multiplicative factor represents the power of the determinant involved.)

$$\begin{aligned} S_{eff}(\phi) &= \frac{i}{2}\mathrm{Tr}\,\ln\left(1 - gG_F(p)\phi(x)\right) \\ &= \frac{i}{2}\mathrm{Tr}\left(-gG_F(p)\phi(x) - \frac{g^2}{2}G_F(p)\phi(x)G_F(p)\phi(x) + \cdots\right) \end{aligned} \tag{10.45}$$

where we have expanded, as is done in derivative expansion, the logarithm in powers of the propagator. To compare with the earlier calculations (see section 1.7) of nonanalyticity in the self-energy, for example, let us consider only the term quadratic in the ϕ field

$$\begin{aligned} S^Q_{eff} &= -\frac{ig^2}{4}\mathrm{Tr}\left(G_F(p)\phi(x)G_F(p)\phi(x)\right) \\ &= S^{Q\,(0)}_{eff} + S^{Q\,(\beta)}_{eff} \end{aligned} \tag{10.46}$$

where we have separated S^Q_{eff} into the zero temperature part and a part that depends explicitly on temperature.

From the structure of the propagator at finite temperature in eq. (10.44), we see that the temperature dependent part of S^Q_{eff} has the explicit form

$$\begin{aligned} S^{Q\,(\beta)}_{eff} = -\frac{ig^2}{4}\mathrm{Tr}\,\Big(&G^\beta_F(p)\phi(x)G^0_F(p)\phi(x) \\ &+G^0_F(p)\phi(x)G^\beta_F(p)\phi(x) \\ &+G^\beta_F(p)\phi(x)G^\beta_F(p)\phi(x)\Big) \end{aligned} \tag{10.47}$$

Once again, we can move all the momentum dependent factors past coordinate dependent ones through the relation

$$\phi(x)f(p) = [f(p - i\partial)\phi(x)] \tag{10.48}$$

where the square bracket represents the action of the derivatives, namely, the derivatives act only on $\phi(x)$. The derivative expansion (*e.g.*, the last relation in eq. (10.8)) is merely a Taylor series expansion of this formula in powers of derivatives. However, a Taylor series expansion is meaningful only if a function is analytic. Since, we do not know *a priori* whether the temperature dependent part of the action will be analytic, it is much more meaningful to use the relation in eq. (10.48).

Using eq. (10.48), we can move all the momentum dependent factors to the left which would yield the temperature dependent part of the quadratic action to be

$$\begin{aligned}
S_{eff}^{Q(\beta)} &= -\frac{ig^2}{4}\mathrm{Tr}\,\Big(G_F^\beta(p)[G_F^0(p-i\partial)\phi(x)]\phi(x) \\
&\quad + G_F^0(p)[G_F^\beta(p-i\partial)\phi(x)]\phi(x) \\
&\quad + G_F^\beta(p)[G_F^\beta(p-i\partial)\phi(x)]\phi(x)\Big) \\
&= -\frac{g^2}{4}\int\frac{d^4p}{(2\pi)^3}\int d^4x\,\Bigg[\delta(p^2-M^2)\frac{n_B(|p^0|)}{(p-i\partial)^2-M^2+i\epsilon}\phi(x) \\
&\quad + \frac{n_B(|p^0-i\partial^0|)}{p^2-M^2+i\epsilon}\delta((p-i\partial)^2-M^2)\phi(x) \\
&\quad - 2i\pi n_B(|p^0|)n_B(|p^0-i\partial^0|)\delta(p^2-M^2) \\
&\quad \times\delta((p-i\partial)^2-M^2)\phi(x)\Bigg]\phi(x)
\end{aligned} \tag{10.49}$$

Here, the derivatives are assumed to act on the first of the two ϕ fields.

The momentum integral in eq. (10.49) can be easily checked to correspond to the integral studied in section 1.7 (in the real time formalism) in connection with the nonanalyticity of the self-energy with the identification of $p \to -i\partial$. In fact, this can be evaluated using the modified Feynman parameterization and it is easy to show that

$$\mathrm{Re}\, S_{eff}^{Q\beta} = -\frac{g^2}{32\pi^2}\int d^4x \left(\int_0^\infty \frac{pdp}{\omega}\frac{n_B(\omega)}{(-\nabla^2)^{\frac{1}{2}}}\mathrm{Re}\,(\ln R)\phi(x)\right)\phi(x) \tag{10.50}$$

where, as before, we have defined

$$\omega = (p^2 + M^2)^{\frac{1}{2}} \tag{10.51}$$

and R is as defined in eq. (9.79)

$$R = \frac{[\partial_\mu\partial^\mu + 2i(\omega\partial_0 + p(-\nabla^2)^{\frac{1}{2}})][\partial_\mu\partial^\mu - 2i(\omega\partial_0 - p(-\nabla^2)^{\frac{1}{2}})]}{[\partial_\mu\partial^\mu + 2i(\omega\partial_0 - p(-\nabla^2)^{\frac{1}{2}})][\partial_\mu\partial^\mu - 2i(\omega\partial_0 + p(-\nabla^2)^{\frac{1}{2}})]} \tag{10.52}$$

As we have discussed earlier, in chapter 9, this expression is manifestly nonanalytic bringing out the nonanalyticity of the self-energy at finite temperature yet in a different fashion. Although we have chosen to study the two point function (self-energy), the cubic, quartic and higher terms in S_{eff} may, in principle, develop similar nonanalyticities at finite temperature. Thus, we see that the derivative expansion breaks down at finite temperature. It is worth noting here, however, that if we had used commutation relations such as in eq. (10.8)

(as opposed to the true commutation relation of eq. (10.48)), to move the momentum dependent terms past the coordinate dependent ones, then, we would have a well defined derivative expansion. This, however, would correspond to Taylor expanding the integrand before evaluating the integral and we can check that this corresponds to taking the limit $\partial_0 = 0$ and then $(-\nabla^2)^{\frac{1}{2}} \to 0$. However, there is no *a priori* reason why this should be the preferred limit. In other words, if the effective action is analytic, it should not matter whether we Taylor expand the integrand or the integral. However, when the effective action is nonanalytic, as is the case at finite temperature, expanding the integrand becomes questionable. We also note that effective potential, which is crucial in the study of symmetry breaking, is defined (see, for example, eq. (6.55)) as the part of the effective action without any derivatives (*i.e.*, by setting $\partial_\mu = 0$). However, as we have just seen, at finite temperature, the effective action is nonanalytic precisely at this point and, consequently, the definition of the effective potential also becomes nonunique. This is an issue that is not completely understood, but it is worth pointing out that most physical examples such as the degenerate electron gas or superconductivity can be best understood from the point of view of an effective action in the limit $\partial_0 = 0$ followed by $(-\nabla^2)^{\frac{1}{2}} \to 0$.

10.4 Anomaly at $T \neq 0$

As we have seen in the last section, the effective action does become nonanalytic at finite temperature. However, there are still quantities of physical interest in a theory that do not show any nonanalyticity. One such quantity is the chiral anomaly which is known

to be at the heart of the solubility of the two dimensional models. In fact, as we have seen earlier in section 6.6 within the context of the Schwinger model, the chiral anomaly is temperature independent. In this section, we will discuss how the anomaly can be derived within the context of derivative expansion and the question of its temperature dependence.

For the sake of comparison with the results of section 6.6, we will also consider here the example of the Schwinger model with a slightly different notation for clarity. Let us consider the Lagrangian density

$$\mathcal{L}_f = \overline{\psi}(i\partial\!\!\!/ + gV\!\!\!/ + \gamma_5 A\!\!\!/)\psi \tag{10.53}$$

where we have represented the vectorial gauge field by V_μ while the external source for the axial vector current is denoted by A_μ. The generating functional, in the presence of this external source, is given by

$$Z[V_\mu, A_\mu] = e^{iS_{eff}(V_\mu, A_\mu)} = \int \mathcal{D}\psi \mathcal{D}\overline{\psi}\, e^{i\int d^2x\, \mathcal{L}_f} \tag{10.54}$$

so that we have

$$\begin{aligned} \left.\frac{\delta S_{eff}}{\delta A_\mu(x)}\right|_{A_\mu=0} &= \langle \overline{\psi}\gamma_5\gamma^\mu\psi\rangle = \langle j_5^\mu(x)\rangle \\ \text{or,}\quad \langle \partial_\mu j_5^\mu(x)\rangle &= \partial_\mu \left.\frac{\delta S_{eff}}{\delta A_\mu(x)}\right|_{A_\mu=0} \end{aligned} \tag{10.55}$$

This defines the anomaly in the conservation of the axial current.

Following the discussion of the last section (see eq. (10.8)), we see that S_{eff} can be identified with

$$\begin{aligned} S_{eff}(V_\mu, A_\mu) &= -i\mathrm{Tr}\,\ln(1 + S_F(p)(g\not{V}(x) + \gamma_5\not{A}(x))) \\ &= -i\sum_{n=0}^{\infty}\mathrm{Tr}\,\frac{(-1)^n}{n+1}(S_F(p)(g\not{V}(x) + \gamma_5\not{A}(x)))^{n+1} \end{aligned} \tag{10.56}$$

The anomaly, as is clear from eq. (10.55), is obtained only from the term that is linear in A_μ. The first term in the series of eq. (10.56) which contains a term linear in A_μ vanishes following the discussion after eq. (10.29) because the momentum integral is odd. Therefore, the only part of S_{eff} that gives rise to the anomaly comes from the quadratic terms of the series (in particular, from the crossed terms) and has the form

$$S_{eff}^{anomaly} = ig\mathrm{Tr}\,(S_F(p)\not{V}(x)S_F(p)\gamma_5\not{A}(x)) \tag{10.57}$$

where we have used the cyclicity of the trace.

At zero temperature, the fermion Greens function is that of a massless fermion and has the form ($i\epsilon$ is assumed)

$$S_F(p) = S_F^0(p) = \frac{\not{p}}{p^2} \tag{10.58}$$

Using this, we can write $S_{eff}^{anomaly}$ as

$$\begin{aligned} S_{eff}^{anomaly} &= ig\mathrm{Tr}\left(\frac{\not{p}}{p^2}\not{V}(x)\frac{\not{p}}{p^2}\gamma_5\not{A}(x)\right) \\ &= -ig\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho)\left(\frac{p_\mu}{p^2}V_\nu(x)\frac{p_\lambda}{p^2}A_\rho(x)\right) \\ &= -ig\mathrm{Tr}\left((\gamma_5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho)\left[\frac{p_\mu}{p^2}\frac{(p-i\partial)_\lambda}{(p-i\partial)^2}V_\nu(x)\right]A_\rho(x)\right) \end{aligned}$$

$$= -ig\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho)\int \frac{d^2p}{(2\pi)^2}d^2x\left[\frac{p_\mu(p-i\partial)_\lambda}{p^2(p-i\partial)^2}V_\nu(x)\right]A_\rho(x) \tag{10.59}$$

where the derivatives are assumed to act on V_ν alone.

The momentum integral, in eq. (10.59), can be done using dimensional regularization in a straightforward manner and leads to (It is worth emphasizing here that the evaluation here corresponds to a gauge invariant regularization.)

$$S_{eff}^{anomaly} = -\frac{g}{2\pi}\int d^2x\left(\epsilon^{\mu\nu}V_\mu A_\nu - (\eta^{\mu\nu}\epsilon^{\lambda\rho}+\eta^{\lambda\rho}\epsilon^{\mu\nu})\frac{\partial_\mu\partial_\lambda}{\Box}V_\nu A_\rho\right) \tag{10.60}$$

so that from eqs. (10.55) and (10.60), we obtain the anomaly in the axial current to be

$$\begin{aligned}\langle\partial_\mu j_5^\mu\rangle &= \partial_\mu\frac{\delta S_{eff}^{anomaly}}{\delta A_\mu}\\ &= -\frac{g}{2\pi}\epsilon^{\mu\nu}\partial_\nu V_\mu + \frac{g}{2\pi}\epsilon^{\mu\nu}\partial_\mu V_\nu\\ &= \frac{g}{\pi}\epsilon^{\mu\nu}\partial_\mu V_\nu\end{aligned} \tag{10.61}$$

This is, indeed, the gauge invariant anomaly in the Schwinger model at zero temperature and here we have derived it from the derivative expansion.

Let us next analyze the anomaly at finite temperature within the framework of the derivative expansion. At finite temperature, the fermion Greens function has the form (see eq. (2.45))

$$\begin{aligned} S_F(p) &= \not{p}\left(\frac{1}{p^2} + 2i\pi n_F(|p^0|)\delta(p^2)\right) \\ &= S_F^0(p) + S_F^\beta(p) \end{aligned} \tag{10.62}$$

where n_F represents the fermionic distribution function

$$n_F(|p^0|) = \frac{1}{e^{\beta|p^0|}+1}$$

Substituting this into eq. (10.57), we see that, in addition to the zero temperature part, the action for the anomaly will now contain a term that is manifestly temperature dependent and is given by (see eq. (10.59))

$$\begin{aligned} S_{eff}^{anomaly(\beta)} &= -ig\,\mathrm{Tr}\,[\gamma_5(S_F^\beta(p)\not{V}(x)S_F^0(p)\not{A}(x) + S_F^0(p)\not{V}(x)S_F^\beta(p)\not{A}(x) \\ &\quad + S_F^\beta(p)\not{V}(x)S_F^\beta(p)\not{A}(x))] \end{aligned} \tag{10.63}$$

We note that there are two kinds of terms – linear and quadratic in $S_F^\beta(p)$. As in eq. (6.102), let us study them separately.

The two terms linear in $S_F^\beta(p)$ can be written as

$$\begin{aligned} \text{Linear}: &= -ig\,\mathrm{Tr}\left((\gamma_5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho)[p_\mu n_F(|p^0|)\delta(p^2)V_\nu\frac{p_\lambda}{p^2}\right. \\ &\quad \left. + \frac{p_\mu}{p^2}V_\nu p_\lambda n_F(|p^0|)\delta(p^2)]A_\rho\right) \\ &= -ig\,\mathrm{Tr}\left((\gamma_5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho)[p_\mu n_F(|p^0|)\delta(p^2)\frac{(p-i\partial)_\lambda}{(p-i\partial)^2}V_\nu\right. \\ &\quad \left. + \frac{p_\mu}{p^2}(p-i\partial)_\lambda n_F(|(p-i\partial)^0|)\delta((p-i\partial)^2)V_\nu]A_\rho\right) \end{aligned}$$

$$= -2ig(\eta^{\mu\nu}\epsilon^{\lambda\rho} + \eta^{\lambda\rho}\epsilon^{\mu\nu}) \int \frac{d^2p}{(2\pi)^2} d^2x \left[\frac{n_F(|p^0|)\delta(p^2)}{(p - i\partial)^2} + \frac{n_F(|(p - i\partial)^0|)\delta((p - i\partial)^2)}{p^2} p_\mu (p - i\partial)_\lambda V_\nu \right] A_\rho \tag{10.64}$$

The temperature dependent terms in the effective action are finite and, therefore, we have evaluated the Dirac trace above using two dimensional identities. The contribution to the anomaly from these terms can be easily obtained by taking the functional derivative with respect to A_ρ and applying ∂_ρ to the resulting expression. Thus, we obtain, from the terms linear in S_F^β,

$$\begin{aligned} \text{Anomaly}: &= -2ig(\eta^{\mu\nu}\epsilon^{\lambda\rho} + \eta^{\lambda\rho}\epsilon^{\mu\nu}) \int \frac{d^2p}{(2\pi)^2} \frac{n_F(|p^0|)\delta(p^2)}{(p - i\partial)^2} \\ &\quad \times (p_\mu (p - i\partial)_\lambda + p_\lambda (p - i\partial)_\mu)\partial_\rho V_\nu \\ &= -2ig \int \frac{d^2p}{(2\pi)^2} \frac{n_F(|p^0|)\delta(p^2)}{(p - i\partial)^2} [-2i(p - i\partial)^2 \epsilon^{\mu\nu} p_\nu] V_\mu \\ &= -4g\epsilon^{\mu\nu} \int \frac{d^2p}{(2\pi)^2} n_F(|p^0|)\delta(p^2) p_\nu V_\mu = 0 \end{aligned} \tag{10.65}$$

In obtaining this result, we have used the substitution

$$p \to -(p - i\partial)$$

in the second integral in eq. (10.64) as well as the two dimensional identity

$$p^\nu \epsilon^{\lambda\rho} p_\lambda \partial_\rho = \epsilon^{\nu\rho}(p^2 \partial_\rho - p \cdot \partial\, p_\rho) \tag{10.66}$$

This shows that the terms linear in S_F^β do not contribute to the anomaly equation. We can, similarly, analyze the terms quadratic in S_F^β in eq. (10.63).

$$\begin{aligned}
\text{Quadratic}: &= -ig\mathrm{Tr}\,\gamma_5 S_F^\beta(p)\not{V} S_F^\beta(p)\not{A} \\
&= -2ig(\eta^{\mu\nu}\epsilon^{\lambda\rho}+\eta^{\lambda\rho}\epsilon^{\mu\nu})\int \frac{d^2p}{(2\pi)^2}d^2x[n_F(|(p-i\partial)^0|) \\
&\quad \times n_F(|p^0|)\delta(p^2)\delta((p-i\partial)^2)p_\mu(p-i\partial)_\lambda V_\nu]A_\rho \qquad (10.67)
\end{aligned}$$

The contribution to the anomaly equation, from the above action, can again be obtained by functionally differentiating eq. (10.67) with respect to A_ρ and applying ∂_ρ to the resulting expression. This yields

$$\begin{aligned}
\text{Anomaly}: &= -2ig(\eta^{\mu\nu}\epsilon^{\lambda\rho}+\eta^{\lambda\rho}\epsilon^{\mu\nu})\int \frac{d^2p}{(2\pi)^2} n_F(|(p-i\partial)^0|) \\
&\quad \times n_F(|p^0|)\delta(p^2)\delta((p-i\partial)^2)p_\mu(p-i\partial)_\lambda\partial_\rho V_\nu \\
&= 2ig\int \frac{d^2p}{(2\pi)^2} n_F(|p^0|)n_F(|(p-i\partial)^0|)\delta(p^2)\delta((p-i\partial)^2) \\
&\quad \times(-i\epsilon^{\mu\nu}p_\mu(p-i\partial)^2)V_\nu \\
&= 0 \qquad (10.68)
\end{aligned}$$

This shows that even the terms in S_{eff} which are quadratic in S_F^β do not contribute to the anomaly equation. Consequently, the axial anomaly is temperature independent as we had seen earlier in chapter 6 and does not show any nonanalyticity.

10.5 References

Aitchison, I and **C. Fraser**, Phys. Lett. **146B** (1984) 63.

Babu, K.S, A. Das and **P. Panigrahi**, Phys. Lett. **B188** (1987) 133.

Babu, K.S, A. Das and **P. Panigrahi**, Phys. Rev. **D36** (1987) 3725.

Das, A and **M. Hott**, Phys. Rev. **D50** (1994) 6655.

Das, A and **A. Karev**, Phys. Rev. **D36** (1987) 623; *ibid* **D36** (1987) 2591.

Karev, A, "Studies in the Derivative Expansion and Finite Temperature Field Theory", Ph. D. Thesis, University of Rochester (1987).

Schakel, A.M.J, "On Broken Symmetries in Fermi Systems", Ph. D. Thesis, Universiteit van Amsterdam (1989).

Chapter 11

Non-Equilibrium Phenomena

Our discussion, so far, has been within the context of phenomena in thermal equilibrium. However, there are several physical systems of interest which need not evolve in thermal equilibrium. Most important among these are systems with dynamical phase transitions. Such non-equilibrium phenomena are of great interest, not only from the point of view of the study of physical systems in condensed matter, but also in the cosmological context of the study of the evolution of our physical universe. The question of formation and growth of domains is as important in these contexts as it has also become in systems in high energy physics. In this chapter, we will discuss one such phenomenon, namely, the disoriented chiral condensates and how the study of such

systems can be accommodated within the framework of discussions in the earlier chapters.

11.1 Disoriented Chiral Condensates

To introduce the concept of the disoriented chiral condensates, let us recall that the strong force among elementary particles is described by the Lagrangian density of Quantum Chromo Dynamics (see eq. (4.18))

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} + i\overline{\psi}^i\gamma^\mu(D_\mu\psi)^i - m\overline{\psi}^i\psi^i \qquad (11.1)$$

where $a = 1, 2, \cdots, 8$ and $i = 1, 2, 3$. Thus, we assume that the gauge fields or gluons belong, as usual, to the adjoint representation of the color symmetry group, $SU(3)$, while the quarks or the fundamental building blocks of matter belong to the fundamental representation of the group. Note, that , here we have included a mass term for the fermions as well. In addition to the color index, the fermions also carry a flavor index which we have suppressed for simplicity. (The mass parameter depends on the flavor index, namely, quarks of different flavors have different masses.)

Quarks, as we see from eq. (11.1), carry a color index or color charge. On the other hand, observed hadrons – mesons and baryons – are believed to be made of quarks in color singlet combinations so that they do not carry any color charge. It is much the same way as atoms are made of charged constituents – protons and electrons

– and yet are charge neutral. In contrast, however, free quarks are not observed in nature unlike free charged particles. This has led to the postulate that the strong force is so strong that the color charge is confined. Namely, the true physical states of QCD (Quantum Chromo Dynamics) are the color singlet hadronic states. There is some evidence for this from lattice studies in QCD which also suggests that the hadronic matter undergoes a transition from the confined phase to a deconfined phase of quarks and gluons beyond a temperature of about 150 MeV. In other words, as the temperature of the heat bath is increased, the thermal motion can be thought of as weakening the binding due to the color force leading to a quark-gluon plasma phase beyond a critical temperature of 150 MeV. It is clear that such a phase must have existed at very early stages of the universe when the temperatures were very high. But what is even more interesting is that, the transition temperature is not extremely high so that such a phase of matter can even be produced in the laboratory and its properties studied. This has led to the Heavy Ion Collision experiments where heavy nuclei (typically gold on gold at RHIC, Brookhaven), with relatively high energy per nucleon, are collided in a small region of space. The high energy involved in these experiments is sufficient to heat up the small region of collision to high temperatures which is expected to produce the quark-gluon plasma phase. However, it is clear that the collision region as well as the time scales involved in these processes are so small, that conditions are likely to change very fast so that it would even be inappropriate to take the equilibrium formalism of statistical mechanics as a first approximation. These are very likely transitions far out of equilibrium and should be properly studied, theoretically, as non-equilibrium phenomena.

Experimentally, on the other hand, it is essential to identify a signal which would clearly show that the quark-gluon plasm phase was, indeed, produced during the collision. The main difficulty in this lies in the fact that, at such high energies, the background is nontrivial and that the most common signals of the quark-gluon plasma phase transitions are easily masked by other nuclear effects. A signal which is free from such effects is known as the disoriented chiral condensates and can be understood as follows. As we have noted earlier, different quark flavors have different mass. Experimentally, we know that, of all the flavors, the "up" and the "down" quark flavors have particularly small dynamical masses

$$m_u \sim 5MeV \qquad\qquad m_d \sim 10MeV \tag{11.2}$$

The other quark flavors have relatively higher values for the dynamical mass. (Quarks also have what is known as a constituent mass which is $\sim 300MeV$.) It is clear, therefore, that if we are interested in the quark-gluon plasma phase transition, then, the energies (temperatures) we are interested in are much higher than the dynamical masses of the "up" and the "down" quarks and, consequently, for all practical purposes, we can take these two quark flavors to be massless. Restricting ourselves to only these two quark flavors, we note that we can write the dynamical Lagrangian density to be

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} + i\overline{\psi}^i\gamma^\mu(D_\mu\psi)^i \tag{11.3}$$

For the rest of our discussions, we will restrict ourselves to only these two quark flavors and, therefore, eq. (11.3) would represent the dynamical Lagrangian density for our purpose. (Once again, we have suppressed the two flavor indices.)

The important thing to note from eq. (11.3) is the fact that the quarks (fermions) are assumed to be massless. As a result, the fermion Lagrangian density has a larger group of global invariance given by $SU_L(2) \times SU_R(2)$. Namely, the Lagrangian density in eq. (11.3) is invariant under the following two sets of global $SU(2)$ flavor transformations

$$\begin{aligned} \delta_L \psi^i &= \frac{i}{4}(\vec{\epsilon}_L \cdot \vec{\tau})(1-\gamma_5)\psi^i \\ \delta_R \psi^i &= \frac{i}{4}(\vec{\epsilon}_R \cdot \vec{\tau})(1+\gamma_5)\psi^i \end{aligned} \tag{11.4}$$

Here $\vec{\tau}$ stands for three Pauli matrices which are the generators of the two $SU(2)$ flavor transformations and $\vec{\epsilon}_{L,R}$ are the global parameters associated with these two transformations. (Under these transformations, the color index does not change.)

The transformations in eq. (11.4) are also known as chiral $SU(2)$ transformations since they involve the γ_5 matrix. Quantum mechanically, however, it is known that these chiral symmetries are broken because of anomalies and the true invariance of the theory is a reduced one

$$SU_L(2) \times SU_R(2) \to SU_V(2) \tag{11.5}$$

The residual vector $SU(2)$ represents the familiar weak isospin symmetry which would be present even for equal, nonzero quark masses. However, the chiral symmetry, as we know, is dynamically broken and the signal for such a breakdown is the appearance of condensates, in our theory, of the form

$$\langle \overline{\psi}\psi \rangle = \langle \overline{\psi}^i \psi^i \rangle \neq 0 \tag{11.6}$$

In other words, we can think of $\langle\overline{\psi}\psi\rangle$ as the order parameter of chiral symmetry breaking. Very roughly speaking, under the $SU(2) \times SU(2)$ transformations of eq. (11.4),

$$\begin{aligned} \delta(\overline{\psi}\psi) &\sim i\overline{\psi}\,(\vec{\epsilon}\cdot\vec{\tau})\,\gamma_5\psi \\ \delta(\overline{\psi}\vec{\tau}\gamma_5\psi) &\sim i\vec{\epsilon}\,\overline{\psi}\psi \end{aligned} \tag{11.7}$$

where we can think of $\vec{\epsilon}$ as the chiral parameter of transformation (It is, in fact, related to $\vec{\epsilon}_L - \vec{\epsilon}_R$.). From the discussion in chapter 6, eq. (6.17), it is now obvious from eq. (11.7) that the chiral symmetry will be broken when the fermion condensate $\langle\overline{\psi}\psi\rangle$ acquires a nonzero value. The associated Goldstone bosons are also known to be the three massless pions (the mass of the pions arise from weak interactions). As the temperature is increased, chiral symmetry is restored and the critical temperature for this phase transition is also estimated to be about 150 MeV. In other words, beyond a temperature of 150 MeV, the fermion condensate has a vanishing vacuum expectation value. It is curious that the critical temperatures for the confining-deconfining transition as well as chiral symmetry restoration are approximately the same. It is not clear, at present, whether there are really two distinct phase transitions, but it is obvious that if we have temperatures high enough to have a quark-gluon plasma phase, then we will necessarily be in a chirally symmetric phase. However, as the small region consisting of the quark-gluon plasma expands, the temperature inside the plasma will decrease and, eventually, chiral symmetry will be broken again. The direction of chiral symmetry breaking, in the isospin space, however, need not be the same as it was originally, before the collision took place. Therefore, if the value of the order parameter is nonzero along the π^0 direction, say, then eventually, when the plasma will hadronize (*i.e.*,

come down to the true vacuum with a lower energy), it will produce a large number of neutral pions and not charged ones. This could lead to spectacular events where a large number of pions correlated in isospin space would come out of a single collision signalling clearly that chiral symmetry restoration and, therefore, a quark-gluon plasma phase was, indeed, achieved. It has already been suggested that events of this kind may already have been experimentally observed (the so called Centauro events), but the situation is not very clear.

The difficulty with this picture is as follows. The direction of chiral symmetry breaking need not be the same in the entire plasma. In fact, it is more likely that there will be domains in the plasma with distinct directions of chiral symmetry breaking. Such domain structures are, indeed, quite common in many condensed matter systems as well as in the study of cosmology. If the domain sizes are small (and, therefore, the energy contained in such regions is small) so that only a few pions are produced upon hadronization, then, such an event can hardly be distinguished from that of a random production of pions in a high energy collision. In fact, if the process proceeds through equilibrium, then simple arguments show that this would, indeed, be the case. On the other hand, for a process far out of equilibrium, as is the case here, the domain sizes can, in fact, grow rapidly so that after hadronization, one can have a clear signal of the quark-gluon plasma phase through the appearance of a large number of correlated pions. We will show that this is, indeed, the case in an example where the rapid cooling of the plasma phase is modelled as a sudden quench below the critical temperature.

As we have already pointed out in chapter 2, the formalism most suitable for study of such non-equilibrium phenomena is within the context of the closed time path formalism. In the next section, we will describe how this method applies to a non-equilibrium situation through a simple example before moving on to the real problem.

11.2 Free Klein-Gordon Theory

Let us, as a warm up example, consider a free Klein-Gordon theory, in $3+1$ dimensions, described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_i^2}{2}\phi^2 + \frac{\delta m^2(t)}{2}\phi^2 \tag{11.8}$$

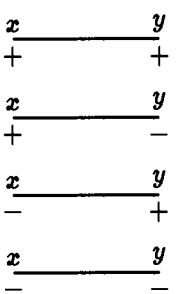
where

$$\delta m^2(t) = \theta(t)\delta m^2 = \theta(t)(m_i^2 + m_f^2) \tag{11.9}$$

This is a time dependent Lagrangian density (and, therefore, the Hamiltonian will also be time dependent) where for $t < 0$, we have a free Klein-Gordon theory with a mass m_i while for positive times, the theory has an imaginary mass (as would be required for a broken symmetry phase described in eq. (6.1)). There is a sudden change of phase at $t = 0$ (prototype of a quenching) and clearly, the evolution of the dynamical system cannot be described through the equilibrium formalism.

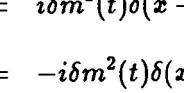
Let us assume that, initially, the system is in equilibrium at a temperature described by β. This system described in eqs. (11.8) and (11.9) is simple enough that one can solve the dynamical equations

explicitly and determine the Greens functions of the theory. In particular, in statistical systems, we are interested in the correlated Greens functions (see eq. (2.39)) which give temperature dependent correlations between various fields. In this section, we will see how the closed time path formalism can be used to derive the same Greens functions. Following our discussion in chapter 2, we note that the Feynman rules for this theory, for $t < 0$, at finite temperature are given by



(11.10)

The interaction vertices of the full theory can be taken to be



(11.11)

As we have noted in eqs. (2.39)-(2.43), the physical Greens functions, G_R, G_A and G_C can be obtained from the Feynman propagators as

$$\widehat{G} = \begin{pmatrix} 0 & G_A \\ G_R & G_C \end{pmatrix} = QGQ^{-1} = Q \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix} Q^{-1} \tag{11.12}$$

where

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$Q^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{11.13}$$

In the present example, we can treat the $\delta m^2(t)$ term in the Lagrangian density, eq. (11.8), as an interaction term and calculate the complete propagator. Or equivalently, let us define

$$\Sigma = \begin{pmatrix} \delta m^2(t) & 0 \\ 0 & -\delta m^2(t) \end{pmatrix} \tag{11.14}$$

and note that the complete propagator is obtained from the inverse

$$\begin{aligned} G &= \left((G^{(0)})^{-1} + \Sigma \right)^{-1} \\ &= \left(1 + G^{(0)}\Sigma \right)^{-1} G^{(0)} = G^{(0)} \left(1 + \Sigma G^{(0)} \right)^{-1} \end{aligned} \tag{11.15}$$

Given this, the complete physical Greens functions are obtained from eqs. (11.12) and (11.15) to be

$$\begin{aligned} \widehat{G} &= QGQ^{-1} \\ &= \left(1 + \widehat{G}^{(0)}\widehat{\Sigma} \right)^{-1} \widehat{G}^{(0)} = \widehat{G}^{(0)} \left(1 + \widehat{\Sigma}\widehat{G}^{(0)} \right)^{-1} \end{aligned} \tag{11.16}$$

where we have defined, consistent with eq. (11.12),

$$\widehat{\Sigma} = Q\Sigma Q^{-1} = \begin{pmatrix} 0 & \delta m^2(t) \\ \delta m^2(t) & 0 \end{pmatrix} \tag{11.17}$$

Let us note that each term in eq. (11.16) consists of a 2×2 matrix which can be simplified to give

$$\widehat{G} = \left[\sum_{n=0}^{\infty} (-1)^n \left(\widehat{G}^{(0)}\widehat{\Sigma}\right)^n\right] \widehat{G}^{(0)} \tag{11.18}$$

Thus, comparing with eq. (11.12), we obtain the complete physical propagators to be

$$\begin{aligned}
G_A &= \sum_{n=0}^{\infty} (-1)^n (G_A^{(0)} \delta m^2(t))^n G_A^{(0)} \\
&= \left(1 + G_A^{(0)} \delta m^2(t)\right)^{-1} G_A^{(0)} = G_A^{(0)} \left(1 + \delta m^2(t) G_A^{(0)}\right)^{-1} \\
G_R &= \sum_{n=0}^{\infty} (-1)^n (G_R^{(0)} \delta m^2(t))^n G_R^{(0)} \\
&= \left(1 + G_R^{(0)} \delta m^2(t)\right)^{-1} G_R^{(0)} = G_R^{(0)} \left(1 + \delta m^2(t) G_R^{(0)}\right)^{-1} \\
G_C &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^n (G_R^{(0)} \delta m^2(t))^m G_C^{(0)} (\delta m^2(t) G_A^{(0)})^{n-m} \\
&= \left(1 + G_R^{(0)} \delta m^2(t)\right)^{-1} G_C^{(0)} \left(1 + \delta m^2(t) G_A^{(0)}\right)^{-1}
\end{aligned} \tag{11.19}$$

Normally, in a relativistic field theory, inversion of factors like the ones in eq. (11.19) are easily achieved by taking the Fourier transformation. Here, however, the two point function, $\delta m^2(t)$, is explicitly time dependent and, consequently, the theory is not time translation invariant and going to the Fourier space does not help. The inversion of the factors, in this case, is slightly more involved, but can be done in the following way. Since time translation is not a symmetry of our

theory, it is more appropriate to work in a mixed coordinate basis where the Greens functions are defined as functions of the time coordinate and the spatial momentum and have the form (See eq. (2.44). These can also be read out from eqs. (2.29) and (2.40).)

$$
\begin{aligned}
G_R^{(0)}(x^0-y^0,\vec{p}) &= \int \frac{dp^0}{2\pi}\,\frac{1}{(p^0+i\epsilon)^2-\omega_i^2}\,e^{-ip^0(x^0-y^0)} \\
&= -\frac{\theta(x^0-y^0)}{\omega_i}\,\sin\omega_i(x^0-y^0) \\
G_A^{(0)}(x^0-y^0,\vec{p}) &= \int \frac{dp^0}{2\pi}\,\frac{1}{(p^0-i\epsilon)^2-\omega_i^2}\,e^{-ip^0(x^0-y^0)} \\
&= \frac{\theta(y^0-x^0)}{\omega_i}\,\sin\omega_i(x^0-y^0) \\
G_C^{(0)}(x^0-y^0,\vec{p}) &= -2i\pi\int \frac{dp^0}{2\pi}\,(1+2n_B(|p^0|))\delta(p^2-m^2)\,e^{-ip^0(x^0-y^0)} \\
&= -\frac{i}{\omega_i}(1+2n_B(\omega_i))\cos\omega_i(x^0-y^0) \qquad (11.20)
\end{aligned}
$$

It is to be noted here that even though the complete Greens functions will not be time translation invariant, the initial theory, for $t < 0$, is time translation invariant and, consequently, the initial physical Greens functions in eq. (11.20) depend only on the difference of the two time coordinates. Furthermore, in writing the Greens functions in eq. (11.20), we have defined

$$\omega_i^2 = \vec{p}^{\,2} + m_i^2 \qquad (11.21)$$

To derive the complete physical Greens functions, let us note next that

$$
\begin{aligned}
(G_R^{(0)}\delta m^2(t))(x^0,y^0;\vec{p}) &= \int \frac{dp^0}{2\pi}\,\frac{e^{-ip^0(x^0-z^0)}}{(p^0+i\epsilon)^2-\omega_i^2}\,\delta m^2\,\theta(z^0)\delta(z^0-y^0) \\
&= \theta(y^0)\int \frac{dp^0}{2\pi}\,\frac{\delta m^2}{(p^0+i\epsilon)^2-\omega_i^2}\,e^{-ip^0(x^0-y^0)}
\end{aligned}
\tag{11.22}
$$

Similarly, we can show that, for $n > 0$,

$$
\begin{aligned}
&\left(G_R^{(0)}\delta m^2(t))(x^0,y^0;\vec{p})\right)^n \\
&\quad = \theta(y^0)\int \frac{dp^0}{2\pi}\left(\frac{\delta m^2}{(p^0+i\epsilon)^2-\omega_i^2}\right)^n e^{-ip^0(x^0-y^0)}
\end{aligned}
\tag{11.23}
$$

so that we have

$$
\begin{aligned}
\left(1+G_R^{(0)}\delta m^2(t)\right)(x^0,y^0;\vec{p}) &= \sum_{n=0}^{\infty}\left(-G_R^{(0)}\delta m^2(t)\right)^n(x^0,y^0;\vec{p}) \\
&= \int \frac{dp^0}{2\pi}\,e^{-ip^0(x^0-y^0)}\left[1+\theta(y^0)\sum_{n=1}^{\infty}\left(-\frac{\delta m^2}{(p^0+i\epsilon)^2-\omega_i^2}\right)^n\right] \\
&= \int \frac{dp^0}{2\pi}\,e^{-ip^0(x^0-y^0)}\left[1-\frac{\delta m^2}{(p^0+i\epsilon)^2-\omega_i^2}\frac{1}{1+\frac{\delta m^2}{(p^0+i\epsilon)^2-\omega_i^2}}\right] \\
&= \int \frac{dp^0}{2\pi}\,e^{-ip^0(x^0-y^0)}\left[1-\theta(y^0)\frac{\delta m^2}{(p^0+i\epsilon)^2-\omega_f^2}\right] \\
&= \delta(x^0-y^0)-\theta(y^0)\delta m^2\int \frac{dp^0}{2\pi}\,\frac{e^{-ip^0(x^0-y^0)}}{(p^0+i\epsilon)^2-\omega_f^2} \\
&= \delta(x^0-y^0)+\theta(x^0-y^0)\theta(y^0)\frac{\delta m^2}{\omega_f}\sin\omega_f(x^0-y^0)
\end{aligned}
\tag{11.24}
$$

where we have defined

$$\omega_f^2 = \omega_i^2 - \delta m^2 \tag{11.25}$$

At this point, we can go ahead and obtain the retarded Greens function defined in eq. (11.20) in a straightforward manner, but we are more interested in the correlated Greens function which we will construct next. We emphasize here, as is clear from eq. (11.24), that the complete retarded function will no longer be time translation invariant. Following the derivation above, we can now show that

$$\begin{aligned}
&\left(1 + \delta m^2(t) G_A^{(0)}\right)^{-1} (x^0, y^0; \vec{p}) \\
&= \int \frac{dp^0}{2\pi}\, e^{-ip^0(x^0-y^0)} \left[1 + \sum_{n=1}^{\infty} \left(-\frac{\delta m^2}{(p^0 - i\epsilon)^2 - \omega_i^2}\right)^n \theta(x^0)\right] \\
&= \int \frac{dp^0}{2\pi}\, e^{-ip^0(x^0-y^0)} \left[1 - \frac{\theta(x^0)\delta m^2}{(p^0 - i\epsilon)^2 - \omega_i^2} \frac{1}{1 + \frac{\delta m^2}{(p^0 - i\epsilon)^2 - \omega_i^2}}\right] \\
&= \delta(x^0 - y^0) - \theta(x^0)\delta m^2 \int \frac{dp^0}{2\pi} \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - i\epsilon)^2 - \omega_f^2} \\
&= \delta(x^0 - y^0) - \theta(y^0 - x^0)\theta(x^0) \frac{\delta m^2}{\omega_f} \sin \omega_f (x^0 - y^0)
\end{aligned} \tag{11.26}$$

We can now obtain the correlated Greens function in a straightforward manner as follows. We note from eq. (11.20) that

$$G_C(x^0, y^0; \vec{p}) = \left[\left(1 + G_R^{(0)} \delta m^2(t)\right)^{-1} G_C^0 \left(1 + \delta m^2(t) G_A^{(0)}\right)^{-1}\right] (x^0, y^0; \vec{p})$$

$$
\begin{aligned}
&= \int dz^0\, dz'^0 \left[\delta(x^0 - z^0) + \theta(x^0 - z^0)\theta(z^0)\frac{\delta m^2}{\omega_f}\sin\omega_f(x^0 - z^0)\right] \\
&\quad \times \left(-\frac{i}{\omega_i}\right)(1 + 2n_B(\omega_i))\cos\omega_i(z^0 - z'^0) \\
&\quad \times \left[\delta(z'^0 - y^0) - \theta(z'^0 - y^0)\theta(z'^0)\frac{\delta m^2}{\omega_f}\sin\omega_f(z'^0 - y^0)\right] \\
&= -\frac{i}{\omega_i}(1 + 2n_B(\omega_i))\left\{\cos\omega_i(x^0 - y^0)\right. \\
&\quad +\theta(y^0)\left(-\cos\omega_i(x^0 - y^0) + \cos\omega_i x^0\cos\omega_f y^0\right. \\
&\quad \left. + \frac{\omega_i}{\omega_f}\sin\omega_i x^0\sin\omega_f y^0\right) \\
&\quad +\theta(x^0)\left(-\cos\omega_i(x^0 - y^0) + \cos\omega_f x^0\cos\omega_i y^0\right. \\
&\quad \left. + \frac{\omega_i}{\omega_f}\sin\omega_f x^0\sin\omega_i y^0\right) \\
&\quad -\theta(x^0)\theta(y^0)\left[-\cos\omega_i(x^0 - y^0) + \cos\omega_f x^0\cos\omega_i y^0\right. \\
&\quad +\frac{\omega_i}{\omega_f}\sin\omega_f x^0\sin\omega_i y^0 + \cos\omega_f y^0(\cos\omega_i x^0 - \cos\omega_f x^0) \\
&\quad \left.\left. + \frac{\omega_i}{\omega_f}\sin\omega_f y^0(\sin\omega_i x^0 - \frac{\omega_i}{\omega_f}\sin\omega_f x^0)\right]\right\}
\end{aligned}
\tag{11.27}
$$

The explicit time dependence of the correlated Greens function is now obvious. Second, we note that, for $x^0 = y^0$, the Greens function becomes

$$G_C = -\frac{i}{\omega_i}(1+2n_B(\omega_i)) \times \left[1+\theta(x^0)\left(-1+\cos^2\omega_f x^0+\left(\frac{\omega_i}{\omega_f}\right)^2\sin^2\omega_f x^0\right)\right] \tag{11.28}$$

We note from eq. (11.28) that, for $x^0 < 0$, the correlated Greens function has the form

$$G_C = -\frac{i}{\omega_i}(1+2n_B(\omega_i)) \tag{11.29}$$

as it should be (see eq. (11.20)). On the other hand, for $x^0 > 0$, the correlated Greens function in eq. (11.28) takes the form

$$\begin{aligned} G_C &= -\frac{i}{\omega_i}(1+2n_B(\omega_i))\left[\cos^2\omega_f x^0+\left(\frac{\omega_i}{\omega_f}\right)^2\sin^2\omega_f x^0\right] \\ &= -\frac{i}{\omega_i}(1+2n_B(\omega_i))\left[1+\frac{1}{2}\left(1-\left(\frac{\omega_i}{\omega_f}\right)^2\right)\left(\cos 2\omega_f x^0-1\right)\right] \end{aligned} \tag{11.30}$$

This is, indeed, exactly what is obtained by solving the differential equations for the Greens functions in the present case. Here, we have tried to derive it from the diagrammatic point of view of the closed time path formalism to bring out the flavor of the calculation for the real problem of the disoriented chiral condensates that we are interested in.

11.3 Nambu-Jona-Lasinio Model

The study of dynamical breaking of chiral symmetry in QCD is nontrivial and, therefore, we choose a simple model to describe the phenomena of chiral symmetry breaking where calculations are at least manageable. The most familiar of such models is known as the Nambu-Jona-Lasinio model and is described by a Lagrangian density

$$\mathcal{L}_{NJL} = i\overline{\psi}^i \not{\partial} \psi^i + \frac{g^2}{4N\Lambda^2}\left[(\overline{\psi}^i\psi^i)^2 - (\overline{\psi}^i\tau^a\gamma_5\psi^i)^2\right] \tag{11.31}$$

Here $i = 1, 2, \cdots, N$ represent the color indices while the two flavor indices of the fermions are suppressed. The three matrices τ^a, $a = 1, 2, 3$, (or $\vec{\tau}$) are the three Pauli matrices which are the generators of the flavor $SU(2)$ symmetry group and which commute with the gamma matrices. In $3+1$ dimensions, the Nambu-Jona-Lasinio theory is nonrenormalizable and we should think of this as an effective low energy theory below the cut-off scale Λ. We will study this model in $\frac{1}{N}$ expansion which is known to yield only planar diagrams at the leading order and nonperturbative results in the coupling constant g. This theory, as can be easily checked, is invariant under the global chiral transformations

$$\begin{aligned} \psi^i &\rightarrow e^{i\vec{\alpha}\cdot\vec{\tau}\gamma_5}\psi^i \\ \overline{\psi}^i &\rightarrow \overline{\psi}^i e^{i\vec{\alpha}\cdot\vec{\tau}\gamma_5} \end{aligned} \tag{11.32}$$

where the three components of $\vec{\alpha}$ represent the constant parameters of transformation. Infinitesimally, these transformations take the form

$$\begin{aligned} \delta\psi^i &= i\vec{\epsilon}\cdot\vec{\tau}\gamma_5\psi^i \\ \delta\overline{\psi}^i &= i\overline{\psi}^i\vec{\epsilon}\cdot\vec{\tau}\gamma_5 \end{aligned} \tag{11.33}$$

We can write the Nambu-Jona-Lasinio model also in an alternate suggestive manner, convenient for our discussion, with the introduction of auxiliary fields as

$$\mathcal{L}_{NJL} = i\overline{\psi}^i \not{\partial} \psi^i - \frac{N\Lambda^2}{2}(\sigma^2 + \pi^a\pi^a) + \frac{g}{\sqrt{2}}\overline{\psi}^i(\sigma + i\gamma_5\pi^a\tau^a)\psi^i \quad (11.34)$$

where we can think of σ and π^a as the scalar and the pseudoscalar fields (The three π^a's can be thought of as corresponding to the three pions). The equations of motion for the auxiliary fields are easily seen to be the constraint equations

$$\begin{aligned} \sigma &= \frac{g}{\sqrt{2}}\overline{\psi}^i\psi^i \\ \pi^a &= \frac{ig}{\sqrt{2}}\overline{\psi}^i\tau^a\gamma_5\psi^i \end{aligned} \quad (11.35)$$

and if we eliminate the auxiliary fields through their equations of motion, eq. (11.35), then, we get back the Lagrangian density of eq. (11.31). In terms of these variables, the infinitesimal chiral symmetry transformations of the theory take the form (in addition to those in eq. (11.33))

$$\begin{aligned} \delta\psi^i &= i\vec{\epsilon}\cdot\vec{\tau}\gamma_5\psi^i \\ \delta\sigma &= 2\vec{\epsilon}\cdot\vec{\pi} \\ \delta\vec{\pi} &= -2\vec{\epsilon}\sigma \end{aligned} \quad (11.36)$$

Although chiral symmetry is a symmetry of the tree level Lagrangian, it is, in fact, dynamically broken. This can be seen in the following way. Let us shift the field variables as

$$\begin{aligned} \sigma &\rightarrow \sigma + \sigma_c \\ \pi^a &\rightarrow \pi^a \end{aligned} \tag{11.37}$$

Namely, we are interested in evaluating the effective potential for the scalar field since we expect the vacuum expectation value of the pseudoscalar fields to vanish. Under this shifting, the Lagrangian density of eq. (11.34) becomes

$$\begin{aligned} \mathcal{L}_{NJL} &= i\overline{\psi}^i \partial\!\!\!/ \psi^i - \frac{N\Lambda^2}{2}(\sigma^2 + \pi^a\pi^a) - N\Lambda^2\sigma_c\sigma - \frac{N\Lambda^2}{2}\sigma_c^2 \\ &\quad + \frac{g\sigma_c}{\sqrt{2}}\overline{\psi}^i\psi^i + \frac{g}{\sqrt{2}}\overline{\psi}^i(\sigma + i\pi^a\tau^a\gamma_5)\psi^i \end{aligned} \tag{11.38}$$

Following the discussion in chapter 6 (see *e.g.*, eqs. (6.69) and (6.75)), we can now obtain the effective potential including the one loop contribution to be

$$\begin{aligned} V_{eff}(\sigma_c) &= V^0(\sigma_c) + V^1(\sigma_c) \\ &= \frac{N\Lambda^2}{2}\sigma_c^2 - 4N\int \frac{d^4k_E}{(2\pi)^4}\ln(k_E^2 + m_{eff}^2) \end{aligned} \tag{11.39}$$

where we have identified

$$m_{eff}^2 = \frac{g^2\sigma_c^2}{2} \tag{11.40}$$

We note here that the factors in front of the second term come from tracing over the color indices, the flavor indices as well as the Dirac indices. Furthermore, the difference in sign from that of eq. (6.69) or (6.75) is because of the fact that for fermions, the power of the determinant is positive (as opposed to bosons where it is negative).

(It is also worth emphasizing that there is a factor of $\frac{1}{2}$ coming from the fact that the fermion two point function is given by $(\not{k} + m_{eff})$ and not $(k^2 - m^2)$ as would be the case for a scalar field.) Finally, we also note that since the Nambu-Jona-Lasinio theory is a low energy effective theory, the momentum integrals should be cut off at Λ. With all this information, we can, of course, evaluate the effective potential as in chapter 6. However, let us note that we are not really interested in the exact form of the effective potential – rather, we would like to know if there exists a nontrivial minimum of the effective potential. Namely, we would like to know if a nontrivial solution of the equation

$$\frac{dV_{eff}}{d\sigma_c} = 0 \tag{11.41}$$

From eqs. (11.39) and (11.40), we see that this is equivalent to

$$N\sigma_c \left[\Lambda^2 - 4g^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + m_{eff}^2}\right] = 0 \tag{11.42}$$

The minimum equation, (11.42), is commonly known as the gap equation and we note here that the integral, in eq. (11.42), can be trivially evaluated with a four-dimensional cut-off. However, we would evaluate it slightly differently so that it can be generalized to finite temperature. Let us first do the k^4 integral in the above followed by the spatial momentum integral which will be cut-off at Λ. In such a case, the gap equation becomes, (k stands for the radial momentum)

$$N\sigma_c \left[\Lambda^2 - 4g^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + m_{eff}^2}}\right] = 0$$

$$\text{or, } \sigma_c \left[\Lambda^2 - \frac{g^2}{\pi^2} \int_0^\Lambda dk \, \frac{k^2}{\sqrt{k^2 + m_{eff}^2}} \right] = 0 \qquad (11.43)$$

The radial integral can be easily done to give

$$\sigma_c \left[\Lambda^2 - \frac{g^2}{2\pi^2} \left\{ \Lambda \sqrt{\Lambda^2 + m_{eff}^2} - m_{eff}^2 \ln \frac{\Lambda + \sqrt{\Lambda^2 + m_{eff}^2}}{m_{eff}} \right\} \right] = 0 \qquad (11.44)$$

We note here that, for large values of Λ, this equation takes the form

$$\sigma_c \left[\left(1 - \frac{g^2}{2\pi^2} \right) \Lambda^2 + \frac{g^4 \sigma_c^2}{4\pi^2} \ln \frac{2\sqrt{2}\Lambda}{g\sigma_c} \right] = 0 \qquad (11.45)$$

We see, from eq. (11.45), that the logarithm is a positive quantity and, consequently, a nontrivial solution of the minimum equation (gap equation) would exist, if

$$\frac{g^4 \sigma_c^2}{4\pi^2} \ln \frac{2\sqrt{2}\Lambda}{g\sigma_c} = - \left(1 - \frac{g^2}{2\pi^2} \right) \Lambda^2 \qquad (11.46)$$

Clearly, this is possible only if

$$- \left(1 - \frac{g^2}{2\pi^2} \right) > 0$$

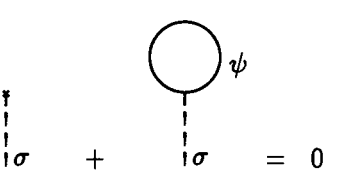
$$\text{or, } \quad g^2 > 2\pi^2 \qquad (11.47)$$

On the other hand, if eq. (11.47) does not hold, then, the minimum of the potential will occur at

$$\sigma_c = 0 \qquad (11.48)$$

In other words, for sufficiently strong coupling, given by eq. (11.47), the σ field picks up a vacuum expectation value which according to eq. (11.35) also implies that the fermion condensate $\overline{\psi}^i\psi^i$ has a nonvanishing vacuum expectation value. Furthermore, from eq. (11.36) as well as our discussions in chapter 6, we see that in this case, chiral symmetry is dynamically broken. On the other hand, for weak coupling (when eq. (11.47) is not true), chiral symmetry is a true symmetry of the theory.

There is an alternate but equivalent method to see this result and goes under the name of vanishing tadpoles. We note that if there is symmetry breaking and we have shifted the fields by the true vacuum expectation value of the field, then, there cannot be a nonvanishing one point function for the scalar field (see *e.g.*, eq. (6.20)). The reason is quite clear. A nonvanishing one point function would imply that the scalar field has a nonvanishing vacuum expectation value which would be inconsistent with the fact that, by shifting, we have defined our quantum theory where the vacuum expectation values of all fields are zero. Thus, we expect that in a quantum theory, all one point functions (or tadpoles) must vanish. If we apply this to the theory in eq. (11.38), then, the one point functions up to one loop must add to zero.



In other words, we must have (remember the trace over the two flavors)

$$
\begin{aligned}
-iN\Lambda^2\sigma_c - \frac{ig}{\sqrt{2}}\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\frac{i}{\not{k} + \frac{g\sigma_c}{\sqrt{2}}} &= 0 \\
\text{or,}\quad -iN\Lambda^2\sigma_c - 4g^2N\sigma_c\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2 - \frac{g^2\sigma_c^2}{2}} &= 0 \\
\text{or,}\quad -iN\sigma_c\left[\Lambda^2 - 4g^2\int \frac{d^4k_E}{(2\pi)^4}\,\frac{1}{k_E^2 + m_{eff}^2}\right] &= 0
\end{aligned}
\tag{11.49}
$$

Thus, we see that the condition for the vanishing of the one point function is identical to the equation for the minimum of the potential, namely, eq. (11.42).

The tadpole method is quite useful if we want to estimate the critical temperature for the restoration of chiral symmetry breaking (in the case that the coupling is strong enough to lead to a breakdown of the chiral symmetry at zero temperature). For example, we note that at finite temperature, there would be an additional temperature dependent term in the second diagram coming from the temperature dependent part of the fermion propagator. From eq. (2.45), we note that this additional term would lead to a contribution, at high temperatures, of the form

$$
\begin{aligned}
&-\frac{ig}{\sqrt{2}}\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\, iS^{\beta}_{++}(k) \\
&\quad = -\frac{ig^2N\sigma_c}{2\pi^3}\int d^4k\, n_F(|k^0|)\delta(k^2 - g^2\sigma_c^2/2) \\
&\quad = -\frac{ig^2N\sigma_c}{2\pi^3}\int d^3k\,\frac{n_F(\omega)}{\omega}
\end{aligned}
$$

$$= -\frac{ig^2 N C^{-1} \sigma_c}{2\pi^3 \beta^2} + O(1/\beta) \tag{11.50}$$

where C is a positive constant whose explicit value is not essential for our discussions. Adding this to the temperature independent part in eq. (11.45), we see that the condition for the vanishing of the tadpoles at finite temperature becomes

$$\sigma_c \left[\left\{ \left(1 - \frac{g^2}{2\pi^2} \right) \Lambda^2 + \frac{g^2 C^{-1}}{2\pi^3 \beta^2} \right\} + \frac{g^4 \sigma_c^2}{4\pi^2} \ln \frac{2\sqrt{2}\Lambda}{g\sigma_c} \right] = 0 \tag{11.51}$$

It is now clear that even if the coupling is strong enough so that eq. (11.47) holds at zero temperature, as the temperature is raised (equivalently, β decreased), beyond a certain temperature, the terms in the curly brackets will become positive and, consequently, at temperatures larger than this critical temperature, the minimum of the potential will occur at $\sigma_c = 0$ (see eq. (11.48)) indicating that chiral symmetry, if it is broken at zero temperature, will be restored beyond this critical temperature. We can determine the critical temperature from eq. (11.51) to be

$$T_c^2 = -2\pi^3 C \left(\frac{1}{g^2} - \frac{1}{2\pi^2} \right) \Lambda^2 \tag{11.52}$$

This shows that the critical temperature for the restoration of chiral symmetry is, in fact, dependent on the magnitude of the coupling constant, namely,

$$T_c = T_c(g) \tag{11.53}$$

Thus, we see that the Nambu-Jona-Lasinio model displays all the features of chiral symmetry breaking and restoration that we expect from QCD. Furthermore, this model is simple calculationally

– at least in $\frac{1}{N}$ expansion – so that we can take it as a toy model for the study of the disoriented chiral condensates. To model the quark-gluon plasma phase and the sudden quenching of the expanding plasma, let us assume the coupling constant of the Nambu-Jona-Lasinio model to have a simple time dependent form

$$g(t) = \theta(t)g \tag{11.54}$$

where we assume that the time independent coupling, g, is strong enough so that the Nambu-Jona-Lasinio model displays chiral symmetry breaking for positive times. We can also assume that the initial temperature of the system is

$$T < T_c(g) \tag{11.55}$$

Thus, our model with the time dependent coupling would, initially ($t < 0$) describe a gas of free massless fermions in a chirally symmetric phase much like the quark-gluon plasma. At $t = 0$, interactions are introduced such that the system suddenly finds itself at temperatures much below the critical temperature. This makes the initial chirally symmetric state unstable and chiral symmetry will be broken completely in parallel with an expanding quark-gluon plasma with a sudden quench. It is worth noting here that since the initial state of the system is chirally symmetric and since the dynamics is also chirally symmetric, $\langle\sigma\rangle$ will have a vanishing average value in the ensemble. However, within any given element of the ensemble, different regions of space will have different directions for symmetry breaking and we are interested in calculating the growth of the correlations of σ with t. We will carry out this calculation in the next section.

11.4 Domain Growth

As we have discussed earlier, we are interested in finding how the size of correlated regions grows with time in the expanding plasma phase. In the present example, this would correspond to determining the time evolution of the σ correlated (or π correlated) regions for positive times. In other words, we are interested in calculating the time dependence of correlated Greens functions, for example, of the kind

$$\langle \sigma(t, \vec{x}) \sigma(t, 0) \rangle$$

This calculation is, therefore, similar to the one we had already performed earlier in connection with the free Klein-Gordon theory. However, here, we are dealing with an interacting theory and, therefore, the calculations will be slightly more involved. First, we note that because of interactions, the σ Greens functions will have radiative corrections of the form

$$G^{-1}_{ab}(x, y) = G^{(0)\,-1}_{ab}(x, y) + \Sigma_{ab}(x, y) \qquad a, b = \pm \qquad (11.56)$$

where the self-energy, Σ_{ab}, can be calculated diagrammatically. Here, we will define Σ_{+-} and Σ_{-+} with a relative minus sign since one of the external legs would correspond to the $-$ branch. With this, then, the physical sigma propagators can be obtained as in eq. (11.16) and we have

$$\Sigma = \begin{pmatrix} \Sigma_{++} & -\Sigma_{+-} \\ -\Sigma_{-+} & \Sigma_{--} \end{pmatrix}$$

$$\begin{aligned}
\hat{\Sigma} &= Q\Sigma Q^{-1} = \begin{pmatrix} \Sigma_C & \Sigma_A \\ \Sigma_R & 0 \end{pmatrix} \\
G_R &= G_R^{(0)}(1 + \Sigma_R G_R^{(0)})^{-1} \\
G_A &= G_A^{(0)}(1 + \Sigma_A G_A^{(0)})^{-1} \\
G_C &= G_R \Sigma_C G_A + (1 + G_R^{(0)}\Sigma_R)^{-1} G_C^{(0)} (1 + \Sigma_A G_A^{(0)})^{-1} \quad (11.57)
\end{aligned}$$

The relevant quantities for our calculation are the Greens functions for the fermions given in eq. (2.45) (with $m = 0$ and the color and flavor diagonal indices suppressed) which would allow us to calculate the self-energy for the σ field. The tree level physical Greens functions for the σ field are also easily obtained to be

$$\begin{aligned}
G_R^{(0)} &= -\frac{1}{N\Lambda^2} \\
G_A^{(0)} &= -\frac{1}{N\Lambda^2} \\
G_C^{(0)} &= 0 \quad (11.58)
\end{aligned}$$

This unusual form of the tree level Greens functions can be understood as follows. The σ field is an auxiliary field and, consequently, does not correspond to real propagating particles which can thermalize. Another way of saying this is to note that we can think of the σ field as a massive field with a mass of the order of the cut-off, Λ, and, therefore, they are Boltzmann suppressed at any temperature much lower than the cut-off scale. We note here that because of the vanishing of $G_C^{(0)}$, the correlated Greens function in eq. (11.57) simplifies and has only one term.

Let us next discuss the strategy for calculating G_C before doing the actual calculation. We note that because $g(t) = \theta(t)g$, the self-energies (Feynman or physical) have the generic form

$$\Sigma(x,y) = \theta(x^0)\overline{\Sigma}(x,y)\theta(y^0) \tag{11.59}$$

It follows, therefore, that all self-energies including Σ_R, Σ_A and Σ_C vanish if $x^0 < 0$ or $y^0 < 0$. (Another way of saying this is that for $t < 0$, we have a free theory and, therefore, there cannot be any self-energy graph.) Furthermore, $\overline{\Sigma}$ is a quantity that is calculated with a constant g, much the same way the equilibrium calculation is done (The equilibrium quantities will depend on the difference in the coordinates.). As a result, evaluation of factors such as $(1+\Sigma_R G_R^{(0)})^{-1}$, which can have nontrivial time dependence, becomes simpler for positive times. Thus, for example,

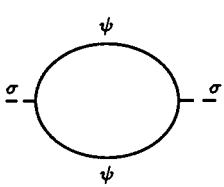
$$\begin{aligned}
(1+\Sigma_R G_R^{(0)})^{-1}(1+\Sigma_R G_R^{(0)}) &= 1 \\
\text{or, } \int d^4z\,(1+\Sigma_R G_R^{(0)})^{-1}(x,z) & \\
\times\left[\delta^4(z-y) - \frac{\theta(z^0)}{N\Lambda^2}\theta(z^0)\overline{\Sigma}_R(z-y)\theta(y^0)\right] &= \delta^4(x-y)
\end{aligned} \tag{11.60}$$

Since, by definition, $\Sigma_R(z-y)$ vanishes for $z^0 < y^0$, if we restrict ourselves to positive times, namely, $y^0 > 0$, the two step functions in the second term can be ignored and the inversion can be carried out much like in the equilibrium case. The knowledge of $(1+\Sigma_R G_R^{(0)})^{-1}(x,y)$, for positive times $x^0, y^0 > 0$ is enough to determine the retarded Greens function as

$$\begin{aligned} G_R(x,y) &= \int d^4z\, G_R^{(0)}(x,z)(1+\Sigma_R G_R^{(0)})^{-1}(z,y) \\ &= \overline{G}_R(x-y) \end{aligned} \tag{11.61}$$

We can easily see that the same argument also goes through for the advanced Greens function. But it is, in general, not true for all the Greens function. In particular, we note here that the Feynman Greens function describes propagation both forwards and backwards in time and, therefore, the determination of the Feynman greens function would be much more involved. As we will see shortly, we only need G_R and G_A for positive times for the purpose of our calculation, we can use

$$\begin{aligned} \overline{G}_R(x,y) &= \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{-N\Lambda^2 + \overline{\Sigma}_R(k)} \\ \overline{G}_A(x,y) &= \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{-N\Lambda^2 + \overline{\Sigma}_A(k)} \\ G_C(x,y) &= \int d^4z d^4z'\, \overline{G}_R(x-z)\Sigma_C(z,z')\overline{G}_A(z'-y) \\ &= \int d^4z d^4z'\, \overline{G}_R(x-z)\theta(z^0)\overline{\Sigma}_C(z-z')\theta(z'^0)\overline{G}_A(z'-y) \end{aligned} \tag{11.62}$$



The self-energy, $\overline{\Sigma}_{ab}$, $a,b = \pm$, can be evaluated diagrammatically, in the $\frac{1}{N}$ expansion where the leading term is the fermion

bubble, with the Feynman rules of the theory and are determined to be

$$\begin{aligned} i\overline{\Sigma}_{ab}(p) &= (ab)(-)\left(\frac{ig}{\sqrt{2}}\right)^2 2N\int\frac{d^4k}{(2\pi)^4}\,iS_{ab}(k)iS_{ba}(k+p) \\ &= (-ab)g^2N\int\frac{d^4k}{(2\pi)^4}\,S_{ab}(k)S_{ba}(k+p) \end{aligned} \tag{11.63}$$

where the indices a, b are not summed. The color and the flavor indices of the propagator are explicitly taken out as well as the negative sign coming from the fermion loop. With these, the self-energy takes the form

$$\begin{aligned} \overline{\Sigma}_{++}(p) &= 4g^2N\int\frac{d^3k}{(2\pi)^3} \\ &\quad\times\left[\left(\frac{1-2n(\omega_k)}{2\omega_k}\frac{p^0\omega_k+\vec{p}\cdot\vec{k}}{(p^0-\omega_k)^2-\omega_{k+p}^2}-\frac{i\pi}{\omega_k}(p^0\omega_k-\vec{p}\cdot\vec{k})\right.\right. \\ &\quad\left.\times\,\delta((p^0+\omega_k)^2-\omega_{k+p}^2)(\frac{\epsilon(p^0)}{2}+n(\omega_k)-n(\omega_k)n(\omega_{k+p})\right) \\ &\quad\left.+\,p^0\to-p^0\right] \\ &= \overline{\Sigma}^*_{--}(p^*) \\ \overline{\Sigma}_{+-}(p) &= -4i\pi g^2N\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\left[(p^0\omega_k-\vec{p}\cdot\vec{k})\delta((p^0+\omega_k)^2-\omega_{k+p}^2)\right. \\ &\quad\times(n(\omega_k)-1)(n(p^0+\omega_k)-\theta(-p^0-\omega_k)) \\ &\quad+(-(p^0)^2-\vec{p}\cdot\vec{k})\delta((p^0-\omega_k)^2-\omega_{k+p}^2) \\ &\quad\left.\times n(\omega_k)(n(p^0-\omega_k)-\theta(-p^0+\omega_k))\right] \end{aligned}$$

$$\begin{aligned}\overline{\Sigma}_{-+}(p) &= -4i\pi g^2 N \int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\Big[(p^0\omega_k - \vec{p}\cdot\vec{k})\delta((p^0+\omega_k)^2-\omega_{k+p}^2) \\ &\times n(\omega_k)(n(p^0-\omega_k)-\theta(-p^0+\omega_k)) \\ &+(-(p^0)^2-\vec{p}\cdot\vec{k})\delta((p^0-\omega_k)^2-\omega_{k+p}^2) \\ &\times(n(\omega_k)-1)(n(p^0+\omega_k)-\theta(-p^0-\omega_k))\Big] \qquad (11.64)\end{aligned}$$

From these, we can construct $\overline{\Sigma}_R$, $\overline{\Sigma}_A$ and $\overline{\Sigma}_C$ (see, for example, eq. (11.57)), which after some simplification take the form

$$\begin{aligned}\overline{\Sigma}_R(p) &= 4g^2 N \int \frac{d^3k}{(2\pi)^3}\frac{1-2\omega_k}{2\omega_k} \\ &\times\left[\frac{p^0\omega_k+\vec{p}\cdot\vec{k}}{(p^0-\omega_k+i\epsilon)^2-\omega_{k+p}^2}+(p^0,\epsilon)\rightarrow(-p^0,-\epsilon)\right] \\ &= \overline{\Sigma}_A^*(p^*) \\ \overline{\Sigma}_C(p) &= 4i\pi g^2 N((p^0)^2-\vec{p}^2)\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\Big[\delta((p^0+\omega_k)^2-\omega_{k+p}^2) \\ &\times(\frac{\epsilon(p^0)}{2}+n(\omega_k)-n(\omega_k)n(p^0+\omega_k))+p^0\rightarrow -p^0\Big] \qquad (11.65)\end{aligned}$$

In these expressions, $\epsilon(p^0)$ represents the alternating step function which can also be identified with $sgn(p^0)$.

We can now obtain $\overline{G}_{R(A)}$ in a straightforward manner from eq. (11.62). Furthermore, if we know the locations of the poles of $\overline{G}_{R(A)}(p^0,\vec{p})$, namely, α_p (α_p^*), then, we can also evaluate the correlated function, $\overline{G}_C$, in a simple manner. The location of the poles can be determined in the following manner. For example, we see from eq. (11.62) that at the location of the pole of $\overline{G}_R$, we have

$$-N\Lambda^2 + \overline{\Sigma}_R(p^0, \vec{p}) = 0 \tag{11.66}$$

where $\overline{\Sigma}_R$ is given in eq. (11.65). Let us Taylor expand this relation about $\vec{p} = 0$ and look at the lowest order term. In other words, let us set $\vec{p} = 0$ in the above equation. In this case, eq. (11.66) becomes

$$\Lambda^2 \left(1 - \frac{g^2}{2\pi^2}\right)$$
$$+\frac{g^2}{8\pi^2}(p^0)^2 \ln\left(\frac{-4\Lambda^2}{(p^0)^2}\right) - \frac{16g^2}{\pi^2}\int_0^\Lambda dk \frac{1}{e^{\beta k}+1}\frac{k^3}{(p^0)^2 - 4k^2} = 0 \tag{11.67}$$

A simple graphical analysis shows that this equation, for $g^2 > 2\pi^2$ and $T < T_c$ always has a solution for $(p^0)^2 < 0$. Let us call this root as $-M^2$. To find the root of eq. (11.66) when the next term in the Taylor expansion is retained, we substitute

$$(p^0)^2 \to -M^2 + b\vec{p}^2 \tag{11.68}$$

and retain terms up second order in $\vec{p}$. Here b is an arbitrary constant to be determined. After some tedious algebra, we obtain that at the location of the pole to this order, we have

$$(\frac{7}{3} - b)I_3 + (\frac{5}{3} - b)I_5 = 0 \tag{11.69}$$

where $I_{3(5)}$ stand for the integrals

$$\begin{aligned} I_3 &= 4M^4 \int_0^\Lambda dk \frac{k^3 \tanh(\beta k/2)}{(M^2+4k^2)^3} \\ I_5 &= 16M^2 \int_0^\Lambda dk \frac{k^5 \tanh(\beta k/2)}{(M^2+4k^2)^3} \end{aligned} \tag{11.70}$$

Once again, with a little bit of analysis, it is easy to see that the solution of eq. (11.69) lies within the range

$$\frac{5}{3} < b < \frac{7}{3} \tag{11.71}$$

The position of the pole to this order can, therefore, be written as

$$\begin{aligned} (p^0)^2 &= \alpha_p^2 = -M^2 + b\vec{p}^2 \\ \text{or,} \quad \alpha_p &\sim \pm i\left(M - \frac{b\vec{p}^2}{2M}\right) \end{aligned} \tag{11.72}$$

where b takes the value as given in eq. (11.71). The exact values of M and b are not important for our discussion. Rather, what is important is the wrong sign of the mass term, $-(M^2)$ which signals the instability of the initial state. Furthermore, we can carry out our discussion to higher orders, but as we will see shortly, this is enough for our discussion. As we have noted earlier, the location of the pole for the advanced Greens function is simply given by the complex conjugate.

We can now calculate, using eq. (11.62), the equal time correlated Greens function, namely, for $x = (t, \vec{x})$ and $y = (t, 0)$ (and $t > 0$)

$$\begin{aligned} G_C(x,y) &= \int d^4z d^4z' \,\theta(z^0)\theta(z'^0) \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \\ &\quad \times e^{-ik\cdot(x-z)-ip\cdot(z-z')-iq\cdot(z'-y)}\, \overline{G}_R(k)\overline{\Sigma}_C(p)\overline{G}_A(q) \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{dp^0}{2\pi}\frac{dq^0}{2\pi}\frac{i}{k^0 - p^0 + i\epsilon}\frac{i}{p^0 - q^0 + i\epsilon} \\ &\quad \times e^{i\vec{k}\cdot\vec{x} - i(k^0 - q^0)t}\, \overline{G}_R(k^0, \vec{k})\overline{\Sigma}_C(p^0, \vec{k})\overline{G}_A(q^0, \vec{k}) \end{aligned} \tag{11.73}$$

There are several things to note here. First of all, it is the limited range of integrations of z^0 and z'^0 which produce the first two factors (as opposed to the usual δ functions which ensure conservation of energy). Second, from eq. (11.72), we note that the system is unstable and, consequently, $\overline{G}_R$ grows with time. As a result, one should be very careful in defining the Fourier transform of such functions. It is well known that, for functions growing with time as $\overline{G}_R \sim e^{\alpha t}$ for large t, the Fourier transform, $\overline{G}_R(\omega)$, will be analytic only above the line Im $\omega = \alpha$ and, consequently, the inverse Fourier transform is defined along a line slightly above Im $\omega = \alpha$. The opposite holds for the advanced Greens function where the region of analyticity is Im $\omega < \alpha$. Keeping these in mind, we can perform the integrations over k^0 and q^0 in eq. (11.73) which yields

$$\begin{aligned}
\int \frac{dk^0}{2\pi} \frac{e^{-ik^0 t}\overline{G}_R(k^0,\vec{k})}{k^0 - p^0 + i\epsilon} &= -i\left(\overline{g}_R(\alpha_k,\vec{k})\frac{e^{-i\alpha_k t}}{\alpha_k - p^0} + e^{-ip^0 t}\overline{G}_R(p^0,\vec{k})\right) \\
\int \frac{dq^0}{2\pi} \frac{e^{iq^0 t}\overline{G}_A(q^0,\vec{k})}{q^0 - p^0 - i\epsilon} &= i\left(\overline{g}_A(\alpha_k^*,\vec{k})\frac{e^{i\alpha_k^* t}}{\alpha_k^* - p^0} + e^{ip^0 t}\overline{G}_A(p^0,\vec{k})\right)
\end{aligned} \tag{11.74}$$

where $\overline{g}_{R(A)}$ represent the residues at the poles α_k (α_k^* respectively).

As we have noted earlier, α_k is complex, leading to both exponentially growing as well as damped behavior. For large times, it is the exponentially growing term which would dominate. Furthermore, we note that the terms independent of α_k do not grow with time and are the ones that would have been present in an equilibrium situation where the coupling constant would be constant. Thus, keeping only the dominant terms, we obtain, for $t > 0$,

$$G_C(t,\vec{x}) = \int \frac{dp^0}{2\pi}\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, |\overline{g}_R(\alpha_k,\vec{k})|^2\, \frac{e^{-i(\alpha_k-\alpha_k^*)t}}{|p^0-\alpha_k|^2}\, \overline{\Sigma}_C(p^0,\vec{k})$$

$$= \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, e^{-i(\alpha_k-\alpha_k^*)t}\, f(\vec{k}^2,T) \qquad (11.75)$$

where

$$\begin{aligned} f(\vec{k}^2,T) &= g^2 N \int \frac{d^3k}{(2\pi)^3}\, \frac{|\overline{g}_R(\alpha_k,\vec{k})|^2}{\omega_k\omega_{k+p}} \\ &\quad \times \left[\frac{((\omega_k+\omega_{k+p})^2-\vec{k}^2)}{|\omega_p+\omega_{k+p}+\alpha_k|^2}\left(\frac{1}{2}-n(\omega_p)+n(\omega_p)n(\omega_{k+p})\right)\right. \\ &+ \frac{((\omega_k-\omega_{k+p})^2-\vec{k}^2)}{|\omega_p-\omega_{k+p}-\alpha_k|^2} \\ &\quad \times \left(\frac{\epsilon(\omega_p-\omega_{k+p})}{2}-n(\omega_p)+n(\omega_p)n(\omega_{k+p})\right) \\ &+ \frac{((\omega_k-\omega_{k+p})^2-\vec{k}^2)}{|\omega_{k+p}-\omega_p-\alpha_k|^2} \\ &\quad \times \left(\frac{\epsilon(\omega_p-\omega_{k+p})}{2}-n(\omega_p)+n(\omega_p)n(\omega_{k+p})\right) \\ &+ \left.\frac{((\omega_k+\omega_{k+p})^2-\vec{k}^2)}{|\omega_p+\omega_{k+p}-\alpha_k|^2}\left(-\frac{1}{2}-n(\omega_p)+n(\omega_p)n(\omega_{k+p})\right)\right] \end{aligned} \qquad (11.76)$$

With a little bit of analysis, it is easy to see that f vanishes for $\vec{k}^2 \to 0$ and, for small $k = |\vec{k}|$, we can write

$$f(\vec{k}^2,T) \to T^3 d(k/T) \qquad (11.77)$$

where $d(x)$ vanishes for $x \to 0$ as a positive power of x, namely, as $\sim x^\alpha$ with $\alpha > 0$. Now, we can evaluate G_C in eq. (11.75) for large positive

times using the saddle point approximation. The sadddle point occurs at

$$k_s = \frac{2iM|\vec{x}|}{bt} \tag{11.78}$$

Therefore, we see that $k \to 0$ as $t \to \infty$ and this justifies our earlier assumption of retaining only the low order terms in the Taylor expansion in eq. (11.67). Using this, we obtain that for large positive times

$$\begin{aligned} G_C(t,|\vec{x}|) &= C(T)\left(\frac{|\vec{x}|}{t}\right)^{\alpha} e^{Mt}\, e^{-\frac{M|\vec{x}|^2}{2bt}} \\ &= C(T)\left(\frac{|\vec{x}|}{t}\right)^{\alpha} e^{Mt}\, e^{-\frac{|\vec{x}|^2}{L^2(t)}} \end{aligned} \tag{11.79}$$

where $C(T)$ is a function independent of $(t,\vec{x})$ and $L(t)$ is the typical size of the domain given by

$$L(t) = \sqrt{\frac{b}{2M}}\, t \tag{11.80}$$

The size of the domain, of course, depends on the details of the theory and is encoded in the values of the parameters b and M. But the power law dependence on t is universal in the approximation that we have used. Equation (11.80) shows that the size of the domains, indeed, grow with time in the quenched approximation and, therefore, if a quark-gluon plasma phase is created and there is sudden quenching, the plasma is likely to produce a large number of coherent pions upon hadronization. We see from our analysis that the domain size appears to increase indefinitely. This is because in the $\frac{1}{N}$ expansion there is no collision. The plasma would thermalize once collisions are taken into account and we have to go beyond the leading order to see that. This analysis shows how the closed time path formalism can be used to study non-equilibrium phenomena.

11.5 References

Anselm, A, Phys. Lett. **B217** (1989) 169.

Anselm, A and **M.G. Ryskin**, Phys. Lett. **B266** (1991) 482.

Bedaque, P.F and **A. Das**, Mod. Phys. Lett. **A8** (1993) 3151 and unpublished notes.

Bjorken, J.D, Int. J. Mod. Phys. **A7** (1992) 4189.

Boyanovsky, D, D.S. Lee and **A. Singh**, Phys. Rev. **D48** (1993) 800.

Nambu, Y and **G. Jona-Lasinio**, Phys. Rev. **122** (1961) 345.

Rajagopal, K and **F. Wilczek**, Nuc. Phys. **B204** (1993) 577.

Schwinger, J, "Lecture Notes of Brandeis University Summer Institute in Theoretical Physics", (1960).

Index

— Symbols —

1PI generating functional, 209
1PI vertex function, 210

— A —

Abelian gauge field, 199
Abelian gauge theory, 106
Abelian Higgs model, 200, 235
Adjoint representation, 107, 110
Advanced Greens function, 57
Analytic continuation, 63
Analyticity, 61
Anomaly, 355
Anomaly at $T \neq 0$, 352
Anti-BRST transformation, 118
Anti-periodicity, 133, 137, 139
Anti-time ordering, 52, 153

— B —

Berezin integration, 136
Bogoliubov transformation, 78, 81, 87, 89, 285, 287, 305
Bosonic oscillator, 79
Branch cut, 29
BRST invariance, 112
BRST transformation, 117

— C —

Catalysis, 274, 284
Causal Greens function, 97
Chiral symmetry breaking, 366, 377
Chiral symmetry restoration, 366, 383
Circled propagator, 155
Circled vertex, 155
Classical field, 206, 214

Classical potential, 205
Closed Time Path formalism, 39, 48, 369
Coherent state, 135
Complex conjugate, 164
Confinement, 363
Conserved charge, 193
Correlated Greens function, 57, 374
Covariant derivative, 109, 200
Covariant description, 32
Critical temperature, 384
Cut diagram, 164
Cut propagator, 165
Cutkosky rule, 150
Cutting description, 163
Cutting rule, 149–150, 165, 187

— D —

Degenerate electron gas, 261
Density matrix, 1, 5, 41
Derivative expansion, 333–334
Derivative expansion at $T \neq 0$, 347
Diagrammatic method, 150
Discontinuity, 27
Disoriented chiral condensate, 362
Dispersion relation, 62, 96, 98
Domain growth, 386
Double well potential, 199
Dynamical mass generation, 283
Dynamical symmetry breaking, 226, 286

— E —

Effective action, 210, 214, 234
Effective potential, 205, 211, 234, 379
Ensemble average, 70
Entropy, 42
ϵ regularization, 266
Euler-Lagrange equation, 195, 207, 345

— F —

Fayet-Illiopoulous term, 310
Fermion condensate, 283, 286, 366
Fermionic boundary condition, 133

Fermionic oscillator, 73
Feynman parameterization, 269
Feynman propagator, 151
Feynman rules, 153
Field strength, 110
Finite temperature subtleties, 259
Flavor symmetry, 277

— G —

Gap equation, 380
Gauge fixing, 113
Gauge independence, 120, 141, 234
Gauge independence of mass, 249
Gauge problem, 234
Gauge theory, 105
Generating functional, 46, 70, 206
Generator of transformations, 194
Ghost Lagrangian, 114
Ghost number, 118
Global invariance, 365
Global phase transformation, 193
Global symmetry, 192
Goldstino state, 309, 318
Goldstone boson, 90, 198, 366
Goldstone field, 100
Goldstone state, 103
Goldstone theorem, 99, 103, 199
Graded algebra, 292
Graded Poincare algebra, 293
Grassmann parity, 83
Gupta-Bleuler condition, 119

— H —

Heisenberg picture, 3, 8
Higgs mechanism, 202

— I —

Imaginary part, 150
Imaginary Time formalism, 1, 6
Interaction picture, 6, 8

— K —

Kaluza-Klein theory, 321
Klein-Gordon theory, 90, 368

KMS condition, 95
KMS relation, 3, 11

— L —

Landau levels, 280
Largest time equation, 159, 172, 184
Legendre transformation, 123
Lie algebra, 106
Local invariance, 105, 201
Local symmetry, 199
Long range correlation, 199
Lorentz invariance, 32, 90

— M —

Mass correction, 19
Massive vector boson, 345
Massless Thirring model, 346
Master equation, 124
Matrix structure, 51
Matsubara formalism, 4, 6
Matsubara frequency, 10, 12
Matthews-Salam determinant, 316, 335
Modified combination formula, 271

— N —

Nambu-Jona-Lasinio model, 377
Nicolai map, 312
Nielsen identities, 233, 240, 245
Nilpotency, 116
Nöether current, 126, 193
Non-Abelian gauge theory, 106
Non-analyticity, 24
Nonanalyticity, 287, 322, 347
Nonequilibrium phenomena, 361
Nonvanishing vacuum energy, 297

— O —

One loop correction, 64
One loop potential, 212
Order parameter, 209, 296, 313

— P —

Partition function, 2, 70, 87, 138
Path Integral formulation, 14
Periodicity for ghost field, 139

Physical Greens functions, 369
Physical mass, 248
Physical state, 363
Physical subspace, 119, 127, 139
Plasmon mass, 26
Polarization tensor, 144
Projection operator, 129
Propagator, 13, 47
Proper vertex, 123

— Q —

Quantum Chromo dynamics, 362
Quark-gluon plasma, 363
Quenching, 368, 385

— R —

Real Time formalism, 39
Renormalization, 218
Retarded Greens function, 57, 97
Retarded propagator, 186
Riemann ζ function, 283
R_ξ gauge, 235

— S —

Schrödinger field theory, 83
Schrödinger picture, 41
Schwinger model, 227, 346, 353
Screening mass, 26
Self-energy, 22
Smallest time equation, 159
Soluble models, 338
Spectral representation, 152, 168
Spontaneous supersymmetry breaking, 298, 308
Spontaneous symmetry breaking, 89, 99, 191, 197, 210, 221
Squeezed state, 78, 288
String theory, 321
Super thermal pair, 319
Super-Jacobi identity, 293
Supersymmetric oscillator, 298
Supersymmetric quantum mechanics, 298, 310
Supersymmetry, 291
Supersymmetry charge, 300
Supersymmetry transformation, 299
Symmetry restoration, 221

— T —

Thermal doubling, 53, 73, 79, 167
Thermal Hilbert space, 78, 306
Thermal index, 175
Thermal operator, 81, 88, 305
Thermal operators, 78
Thermal propagator, 91–92
Thermal transformation, 77
Thermal vacuum, 70, 72, 75, 80, 87–88, 101, 285, 304
Thermofield dynamics, 69, 284, 304
Tilde conjugation, 83, 85
Tilde system, 71, 74, 80, 90
Tilde variable, 304
Time translation invariance, 371
Tree level potential, 195, 206

— U —

Ultraviolet divergence, 55
Unitarity, 125, 131, 149
Unitary gauge, 203
Unitary transformation, 76

— V —

Vacuum expectation value, 197, 209, 247
Vanishing tadpoles, 382

— W —

Ward identity, 120, 143, 240
Wess-Zumino model, 323
Witten index, 297, 307

— Y —

Yang-Mills theory, 106